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Final Technical Report
September 1976

RELIABILITY ACCEPTANCE SAMPLING PLANS BASED UPON PRIOR DISTRIBUTION
Implications and Determination of the Prior Distribution

Syracuse University

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This report provides all the necessary information to make practical application of acceptance tests based on a prior distribution, when an exponential distribution of failures and an inverted gamma prior distribution are assumed. Volume I, "Introduction and Problem Definition," presents an introduction to the problem and a summary of the other volumes. Volume II, "Risk Criteria and Their Interpretation," defines the different statistical risks which can be used to characterize the tests and provide a guide for selecting appropriate risks in various practical situations. Volume III, "Implications and

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Determination of the Prior Distribution," provides the means for determining the parameters of the prior distribution from existing data, and discusses the reason for using an inverted gamma. Volume IV, "Design of Testing Plans," provides instructions for establishing a test time and number of allowable failures based on the prior distribution and the selected risks. Volume V, "Sensitivity Analyses," shows the effects on the test parameters caused by changes in the prior parameters.

Block 7 (Continued)

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PREFACE

This report is one of a set of five presenting the results of part of the work done under contract number F-30602-71-C-0312. The report is delivered to RADC in accordance with item A006 of the Contract Data Requirement List. Sponsorship and technical direction of this task originated in the Reliability and Maintainability Engineering Section (A. Coppola, Chief), Reliability Branch (D. Barber, Chief), within the Reliability and Compatibility Division (J. Naresky, Chief) of the Rome Air Development Center. Mr. Anthony Coppola was the Project Engineer who was technically supported by Mr. Jerome Klion.

The titles of the reports on the subject "Reliability Acceptance Sampling Plans Based Upon Prior Distribution" are as follows:

Volume I.	Introduction and Problem Definition.
Volume II.	Risk Criteria nd Their Interpretation.
Volume III.	Implications and Determination of the Prior Distribution.
Volume IV.	Design of Testing Plans.
Volume V.	Sensitivity Analyses.

ABSTRACT

This report deals with the determination of prior density for the case when the parameter of the failure distribution is considered to be random. First, the implications of considering the parameter as a random variable on some aspects of reliability analysis are examined. The effect of the population heterogeneity model on acceptance sampling and the effect of individual heterogeneity model on failure distribution, acceptance sampling, burn-in and system reliability are considered. The determination of the functional form of the prior distribution and the reasons for choosing a conjugate prior are discussed. Methods are given for estimating the prior parameters based upon data from various sampling situations. A comprehensive analysis of some observed failure data, including a study of the efficiency of the estimators, is also presented.

ACKNOWLEDGMENTS

The authors are thankful to Anthony Coppola, Anthony Feduccia and Jerome Klion of RADC for their helpful suggestions. We are particularly indebted to Donald Wu for his valuable programming help and other assistance during the course of this project.

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1. INTRODUCTION

Consider a sequence of electronic components having an exponential failure density given by $f(t|\theta) = \theta^{-1} \exp(-t/\theta)$; $t \geq 0$, $\theta > 0$. Due to the inherent variability in the production process, the mean time to failure θ, which characterizes each component, will differ from component to component. For some production processes, the variation in θ may be so small as to be practically negligible and all components may be assumed to have a constant θ. Other processes may exhibit a slowly drifting tendency such that components produced closer together in time have the same θ where as components produced farther apart in time have different values of θ. In a very general setting, the sequence θ^i, $i = 1,2,\ldots$ can be considered to form a correlated time series with possible non-stationarity and seasonality and must be construed as one realization of the underlying stochastic process for θ. One such sequence of θ's is shown in Figure 1.1. In this report, however, we will only consider the case of independent but varying θ's.

At this point, it is necessary to distinguish between a component, a system and a lot in the context of reliability. A component is a device which is not repaired upon failure such as a light bulb or a vaccuum tube. A system consists of a large number of components and we assume that upon failure, can be repaired to be as good as new. A lot consists of a large number of components. Usually components produced within a certain time span are grouped together in a lot.

Now we discuss two types of models for describing variations in θ.

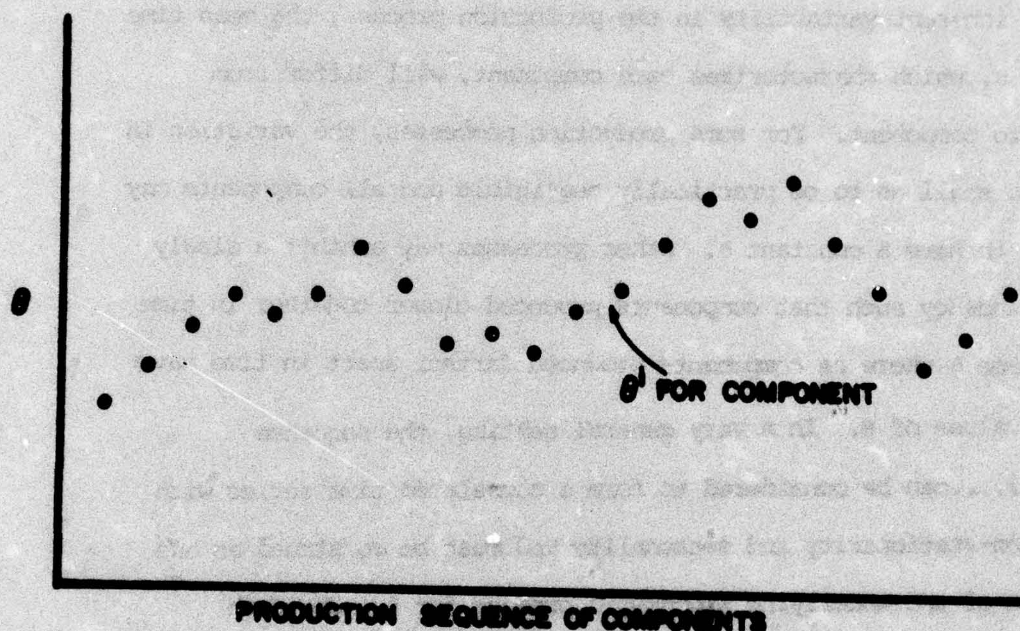

Figure 1.1: A Realization of the Correlated Sequence of θ^i's of the Components.

1.1. Population Heterogeneity Model

(a) For the case of a system, in this model, θ is considered to be a random variable with a frequency distribution $g(\theta)$ and successive systems have different, constant, but unknown, θ^i, $i = 1,2,\ldots$ drawn independently from $g(\theta)$.

(b) For the case of a lot, it is assumed that all the components constituting a lot have identical values of θ. Different lots are taken to have different θ^i, $i = 1,2,\ldots$drawn independently from $g(\theta)$ as illustrated in Figure 1.2.

1.2. Individual Heterogeneity Model

(a) For the case of a lot,the components in each individual lot have different θ^i, $i = 1,2,\ldots$drawn independently from $g(\theta)$. The lots are assumed to have identical $g(\theta)$. This case is illustrated in Fig. 1.3.

(b) A generalization of 1.2 (a) is to assume different $g(\theta)$ for different lots as shown in Fig. 1.4.

Briefly, population heterogeneity model assumes that each population (lot or a system) is homogeneous within itself with a constant θ^i and θ^i's for different populations can be described by a probability density function $g(\theta)$. Individual heterogeneity model assumes that each population has a distribution of individual θ's given by $g(\theta)$. In either case, θ is considered to be a random variable with a distribution $g(\theta)$ called the prior distribution or the compounding distribution.

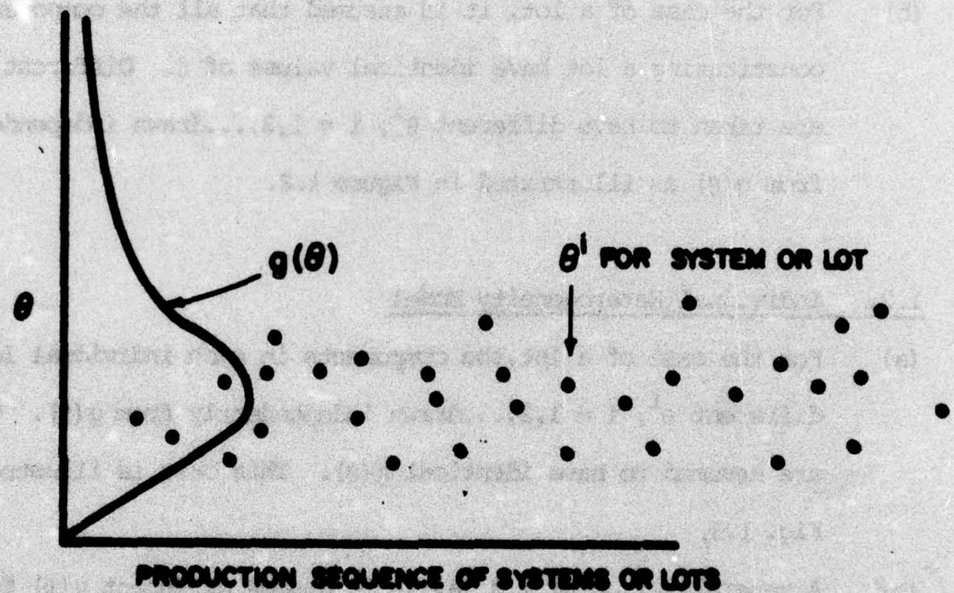

Figure 1.2: A Realization of the Independent Sequence of θ^i's for Systems or Lots

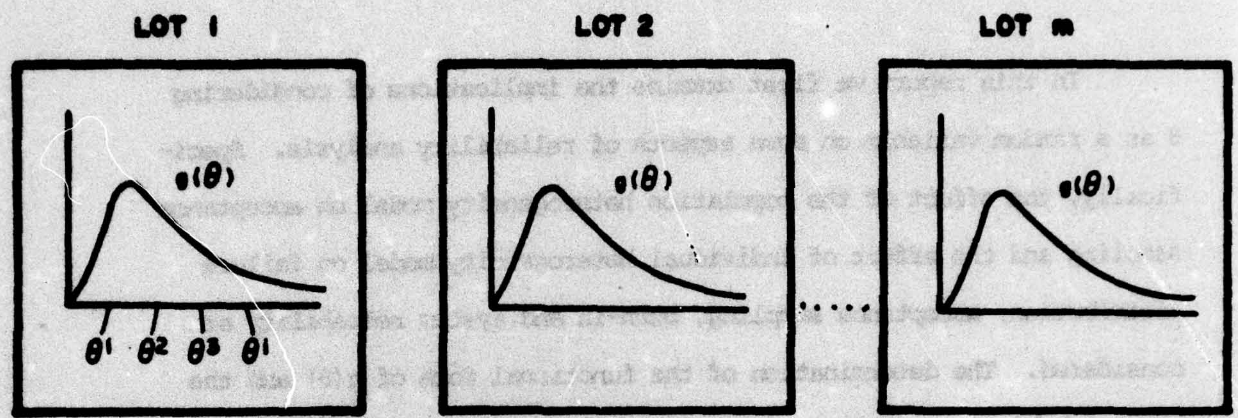

Figure 1.3 A Realization of θ^i's in Different Lots Each With Same $g(\theta)$

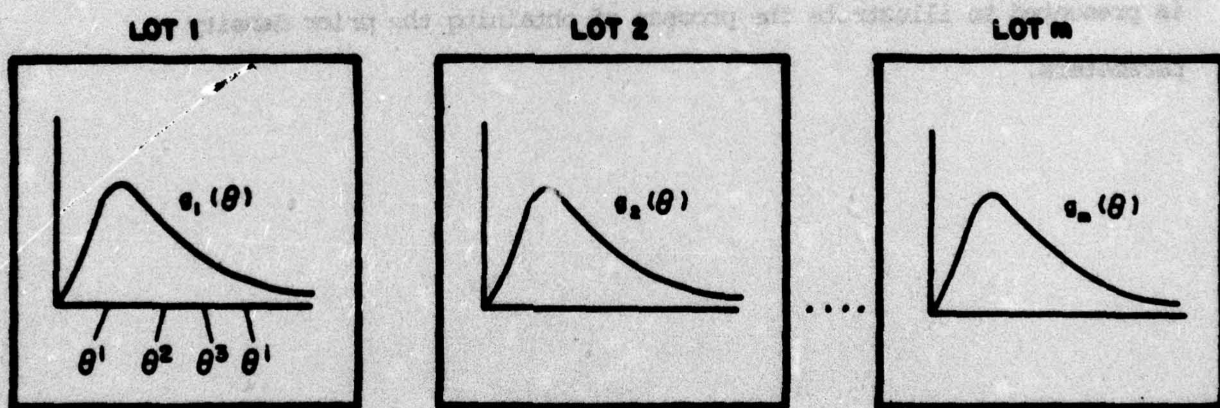

Figure 1.4 A Realization of θ^i's in Different Lots Each With Different $g(\theta)$

In this report we first examine the implications of considering θ as a random variable on some aspects of reliability analysis. Specifically, the effect of the population heterogenaity model on acceptance sampling and the effect of individual heterogenaity model on failure distribution, acceptance sampling, burn-in and system reliability are considered. The determination of the functional form of $g(\theta)$ and the reasons behind a specific choice of inverted gamma prior density are discussed next. Methods for determining the prior parameters based upon data obtained from single sample truncated, censored, with replacement and without replacement acceptance sampling plans for systems and lots are given. A detailed analysis of some observed failure data is presented to illustrate the process of obtaining the prior density parameters.

2. SOME IMPLICATIONS OF PRIOR IN RELIABILITY

In this section we consider how some of the reliability analyses are influenced by treating θ as a random variable instead of a fixed but unknown constant.

2.1. Population Heterogeneity Model

Acceptance Sampling and Demonstration:

In reliability acceptance sampling, we are concerned with making accept/reject decisions regarding a sequence of lots or systems on the basis of data subject to random fluctuations. A definite measure of loss is associated with each of the two decisions when they are inappropriately taken. Finally, each lot is sentenced on its own merit without regard to decisions taken about other lots, i.e. the sequence θ^i, $i = 1,2,...$ is considered to be an independent sequence. In reliability demonstration, one is primarily concerned with determining whether the single system or lot submitted for testing meets the specified reliability requirements and the concept of a sequence of lots is not necessary.

In the context of acceptance sampling and demonstration, $g(\theta)$ may be interpreted as a frequency distribution or as a degree of belief distribution. The two interpretations lead to philosophically different results. A very brief discussion of the two interpretations is given below. For a detailed discussion, the reader is referred to Barnett (1973).

Frequency Interpretation:

The classical acceptance sampling plans do not properly consider the relative frequencies with which different values of θ occur. This disadvantage can be removed by incorporating $g(\theta)$ in the design of acceptance sampling plans. As an example consider the design of single sample truncated plan for a system. A system is put on a life test of duration T and if the observed number of failures r is less than or equal to the acceptance number r* the system is accepted; otherwise, it is rejected. T and r* are determined by limiting the risks of wrong decisions to α and β i.e. producer's risk, $P(R|\theta = \theta_0) \leq \alpha$ and consumer's risk $P(A|\theta = \theta_1) \leq \beta$, where θ_0 is the specified value of θ, θ_1 the minimum acceptable value of θ, and A and R denote acceptance and rejection respectively.

This approach does not take into account any prior information available regarding θ. For example, if it is known a priori that $P(\theta = \theta_1 - \Delta\theta) = 1$ then all systems should be rejected. However, the classical (α,β) plan will accept 100 β% of the systems. A solution is to redefine the risk criteria using $g(\theta)$. As an example consider:

Producer's Risk, $P(R) = \int_\theta P(R|\theta)\, g(\theta)\, d\theta = \dfrac{\text{Number of systems rejected}}{\text{Total Number of Systems Tested}}$

and $\qquad$ (1)

Consumer's Risk, $P(\theta \leq \theta_1|A) = \dfrac{\int_0^{\theta_1} P(A|\theta)\, g(\theta)\, d\theta}{P(A)}$

$$= \frac{\text{Number of accepted systems with } \theta \leq \theta_1}{\text{Number of accepted systems}} = \beta^* \quad (2)$$

The risks P(R) and β^* are defined in the long run average sense giving due weight to the frequency of occurrence of θ. In this case if $P(\theta\leq\theta_1) = 1$, then $P(\theta\leq\theta_1|A) = 1$ and for any specification of $(P(R),\beta^*)$, a test plan does not exist. Similarly if $P(\theta<\theta_1) \leq \beta^*$ then $P(\theta\leq\theta_1|A) \leq \beta^*$ and all the systems are accepted without testing.

Degree of Belief Interpretation:

Consider the i^{th} system under acceptance test. The system has $\theta = \theta^i$ which is an unknown but fixed constant. In the Bayesian formulation, information regarding θ^i may be quantified by the subjective prior $g(\theta^i)$. If the only available information is that θ^i is independently drawn from $g(\theta)$, where $g(\theta)$ is the prior distribution based on a frequency interpretation, then $g(\theta^i) \equiv g(\theta)$. However, in general, $g(\theta^i) \not\equiv g(\theta)$. In fact the notion of $g(\theta)$ as a frequency distribution is dispensable and one may use a conjugate prior to represent $g(\theta^i)$.

If the system is put on a life test and the failure times $t_1, t_2 \ldots t_n$ are observed, then

$$f(t_1, t_2 \ldots t_n|\theta^i) = \frac{1}{\theta^{i^n}} \exp\left(-\sum_{j=1}^{n} t_j/\theta^i\right) \quad (3)$$

When regarded as a function of θ^i, it is known as the likelihood function of θ^i and contains all information regarding θ^i contained in the sample i.e.

$$\ell(\theta^i|t_1\ldots t_n)=\frac{1}{\theta^{i^n}}\exp\left(-\sum_{j=1}^{n}t_j/\theta^i\right) \tag{4}$$

Using Bayes' theorem, the posterior distribution of θ^i given the data is

$$g(\theta^i|t_1\ldots t_n)=\frac{\ell(\theta^i|t_1\ldots t_n)\ g(\theta^i)}{f(t_1\ldots t_n)} \tag{5}$$

If $g(\theta^i)$ is an inverted gamma distribution with parameters (γ,λ), i.e.

$$g(\theta^i)=\frac{\gamma^\lambda}{\Gamma(\lambda)}\,\theta^{i^{-(\lambda+1)}}\,e^{-\gamma/\theta^i}, \tag{6}$$

then $f(t_1\ldots t_n)=\int_{\theta^i} f(t_1\ldots t_n|\theta^i)\ g(\theta^i)\ d\theta^i$

$$=\frac{\gamma^\lambda}{\Gamma(\lambda)}\ \frac{\Gamma(n+\lambda)}{(\Sigma t_j+\gamma)^{n+\lambda}} \tag{7}$$

Hence:

$$g(\theta^i | t_1 \ldots t_n) = \frac{(\sum_{j=1}^{n} t_j + \gamma)^{n+\lambda}}{\Gamma(n+\lambda)} \theta^{i^{-(n+\lambda+1)}} e^{-(\sum_{j=1}^{n} t_j + \gamma)/\theta^i} \tag{8}$$

Equation (8) represents the density of an inverted gamma distribution, i.e.

$$g(\theta^i | t_1 \ldots t_n) \sim Ga' \ (\Sigma t_j + \gamma,\ n+\lambda).$$

The design of a sampling plan can be accomplished by computing appropriate posterior risks or by minimizing a suitable cost function using the posterior density.

Differences in the Two Approaches:

The above two approaches are philosophically different. The first employs $g(\theta)$ to define long run average risks and is meaningful only while dealing with a large sequence of systems or lots (acceptance sampling). The second considers the prior as a subjective distribution and the approach is meaningful in a Bayesian sense whether one is dealing with a single system (demonstration) or a sequence of systems (acceptance sampling).

2.2 Individual Heterogeneity Model

Failure Distribution:

Under the assumption of individual heterogeneity, a lot consists of components with different θ^i, $i = 1,2,\ldots$ drawn independently from $g(\theta)$. If a component is randomly selected from this lot, its life time distribution $f(t)$ is

$$f(t) = \int_\theta f(t|\theta)\; g(\theta)d\theta \qquad (9)$$

Assuming an exponential conditional failure density and $Ga'(\gamma,\lambda)$ prior, we have

$$f(t) = \int_\theta \frac{1}{\theta}\, e^{-t/\theta}\, \frac{\gamma^\lambda}{\Gamma(\lambda)}\, \theta^{-(\lambda+1)} e^{-\gamma/\theta} d\theta = \frac{\lambda\gamma^\lambda}{(\gamma+t)^{\lambda+1}}\; ;\; t \geqslant 0 \qquad (10)$$

i.e. $t \sim$ Pareto (γ,λ). Hence the observed lifetimes $t_1, t_2 \ldots$ will form independent samples from Pareto (γ,λ) distributions.

Some properties of $f(t)$ in Equation (10) are:

$$E(t) = \gamma/(\lambda-1), \quad \lambda > 1 \qquad (11)$$

$$Var(t) = \lambda\gamma^2/(\lambda-1)^2\;(\lambda-2);\; \lambda > 2 \qquad (12)$$

$$R(t) = \left(\frac{\gamma}{\gamma+t}\right)^\lambda \qquad (13)$$

$$h(t) = \lambda/(\gamma+t) \qquad (14)$$

Note that Pareto has a decreasing hazard rate whereas exponential has a constant hazard rate.

Thus, if the individual heterogeneity model applies, the acceptance sampling plans should be based upon the Pareto distribution rather than the exponential distribution.

Burn-in

If components from a heterogeneous lot are put on a burn-in test of duration T, the weak individuals (components with small θ) will be eliminated and the population hazard rate will decrease. Specifically, from Eq. (14), $h(0) = \lambda/\gamma$ and $h(T) = \lambda/(\gamma+T)$. With individual heterogeneity, burn-in will improve reliability. For the case of population heterogeneity, burn-in has no desirable effect.

Let us suppose that the components are required to meet a reliability requirement R for a mission time T_1. The necessary burn-in time may be computed as follows:

$$R(T+T_1|t > T) = \frac{R(T+T_1)}{R(T)} = \left(\frac{\gamma+T}{\gamma+T+T_1}\right)^{\lambda} = R \qquad (15)$$

Hence

$$T = \frac{(\gamma+T_1)\quad R^{1/\lambda}-\gamma}{1 - R^{1/\lambda}} \qquad (16)$$

The expected fraction of parts lost during burn-in is

$$1 - R(T) = 1 - \left(\frac{\gamma}{\gamma+T}\right)^{\lambda} .$$

Optimum burn-in time can be determined on the basis of cost considerations. The cost model will include the cost of burn-in (use of burn-in facility and the cost of components lost) and the savings due to increased population reliability.

System Reliability:

Consider the case of n components in series. The system is unsuccessful if any one of the components fails. Under the usual exponential assumption the system reliability is:

$$R_s(t) = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} e^{-t/\theta} = e^{-nt/\theta} \tag{17}$$

However, for the case of population heterogeneity, with an inverted gamma for θ, the system reliability is given by

$$R_s(t) = \underset{\theta}{E}\,(e^{-nt/\theta}) = \left(\frac{\gamma}{\gamma+nt}\right)^{\lambda} \tag{18}$$

For the case of individual heterogeneity, the system reliability is

$$R_s(t) = \prod_{i=1}^{n} R_i(t) = \prod_{i=1}^{n} \left(\frac{\gamma}{\gamma+t}\right)^{\lambda} = \left(\frac{\gamma}{\gamma+t}\right)^{n\lambda} \tag{19}$$

If there are N different types of components and there are n_i components of the i^{th} type, then

$$R_s(t) = \prod_{i=1}^{N} \left(\frac{\gamma_i}{\gamma_i+t}\right)^{\lambda_i n_i} \tag{20}$$

where γ_i and λ_i are the parameters of the distribution of θ^i for the ith type of component.

For a parallel system, system failure occurs if and only if all components fail. Equations corresponding to (17), (18), (19) and (20) are given by the following Equations (21), (22), (23) and (24) respectively.

$$R_s(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t)) = 1 - (1-e^{-t/\theta})^n \qquad (21)$$

$$R_s(t) = 1 - E_{\theta} (1-e^{-t/\theta})^n$$

$$R_s(t) = 1 - \{1 - (\frac{\gamma}{\gamma+t})^{\lambda}\}^n, \qquad (22)$$

$$R_s(t) = 1 - \prod_{i=1}^{N} \{1 - (\frac{\gamma_i}{\gamma_i+t})^{\lambda_i}\}^{n_i} \qquad (23)$$

Similar expressions may be obtained for other types of systems.

3. DETERMINATION OF PRIOR DENSITY

The choice of the functional form and parameter values of the prior density is critical since a wrong choice of prior will seriously vitiate subsequent analysis. Information regarding the frequency distribution $g(\theta)$ may be available from the following sources.

- Field (operational) failure data
- Failure data from previous acceptance tests.
- Failure data on similar components or systems
- Subjective evaluation based upon a knowledge of system design, failure data from design, development, prediction, assessment and demonstration phases etc.

Subjective information regarding the frequency distribution $g(\theta)$ may be quantified using the 'betting odds' approach. Alternately, if the form of the prior density is known, the parameter values may be 'guessed'.

The problem of empirical determination of prior is somewhat involved since the random variable θ cannot be directly observed. The failure data from which the prior must be obtained is usually available in the following two forms: For a system the data (T_i, x_i), $i = 1,2,\ldots n$, are available where T_i and x_i represent, respectively, the duration of life test and the observed number of failures for the i^{th} system and n is the number of similar systems on a life test. For a lot (n_i, T_i, x_i), $i = 1,2,\ldots N$ are available where N is the number of lots, n_i is the number of components from the i^{th} lot on test for duration T_i and x_i is the number of failures in T_i. In each case, individual times to failure may be available.

Thus, the available data generally consists of observations on some random variable Y, which may be the number of failures, sample MTBF, time to failure etc., and not from the distribution of θ. Hence the marginal distribution of y,

$$f(y) = \int_0^{\infty} f(y|\theta)\, g(\theta)d\theta \qquad (25)$$

must be used to estimate $g(\theta)$.

It should be noted that $g(\theta)$ is a compounding distribution in the commonly used terminology (Ord, 1972). However, in the current context $g(\theta)$ is not only the compounding distribution but also represents our prior knowledge about θ. Mathematically, there is no difference and results of compound distributions apply. However, to properly reflect the role of $g(\theta)$, it is called the prior distribution of θ.

Suppose $g_1(\theta)$ and $g_2(\theta)$ are two priors for θ. For using $f(y)$ to estimate $g(\theta)$ it is necessary to ensure that if the following relationship holds,

$$f(y) = \int_0^{\infty} f(y|\theta)\, g_1(\theta)d\theta = \int_0^{\infty} f(y|\theta)\, g_2(\theta)d\theta \qquad (26)$$

than $g_1(\theta) = g_2(\theta)$.

This identifiability of prior is crucial. In this study it is ensured by results due to Teicher (1960, 1961).

Given the failure data, the problem of estimating $g(\theta)$ is very complicated if the form of the prior density is unspecified and the data comes from unplanned experiments. In the following we assume that $f(t|\theta)$ is negative exponential, $g(\theta)$ is inverted gamma with parameters (γ,λ) i.e. $g(\theta) \sim Ga'(\gamma,\lambda)$ and (γ,λ) are estimated from planned experiments.

3.1. Inverted Gamma Distribution

The inverted gamma prior density is given by

$$g(\theta) = \frac{\gamma^{\lambda}}{\Gamma(\lambda)} \theta^{-(\lambda+1)} e^{-\gamma/\theta} \qquad \gamma, \lambda, \theta > 0 \tag{27}$$

$g(\theta)$ is chosen to be of this form because (also see Schafer et. al. (1970)):

(a) Subsequent data analysis shows $Ga'(\gamma,\lambda)$ to be an adequate prior.

(b) As seen from Figures 2.1 and 2.2, inverted gamma is very flexible and is capable of representing many practical situations.

(c) Being a conjugate density, it implies mathematical ease.

Inverted gamma is a skewed distribution and all moment up to $[\lambda]$ exist. The median and mode always exist. Some of its properties are:

Mean, $\mu_p = \gamma/(\lambda-1)$ (28)

Variance, $\sigma_p^{\,2} = \gamma^2/(\lambda-1)^2\ (\lambda-2)$ (29)

Mode, $\gamma/(\lambda+1)$ (30)

3.2. Parameter Estimation from Planned Experiments

We now consider the estimation of the prior parameters γ and λ based on failure data obtained from planned experiments. Specifically, failure data arising from truncated, censored, with replacement and without replacement single sample plans for systems and lots are considered. In each case, the problem is reduced to one of fitting the appropriate marginal densities to the observed data. The marginal densities for the various

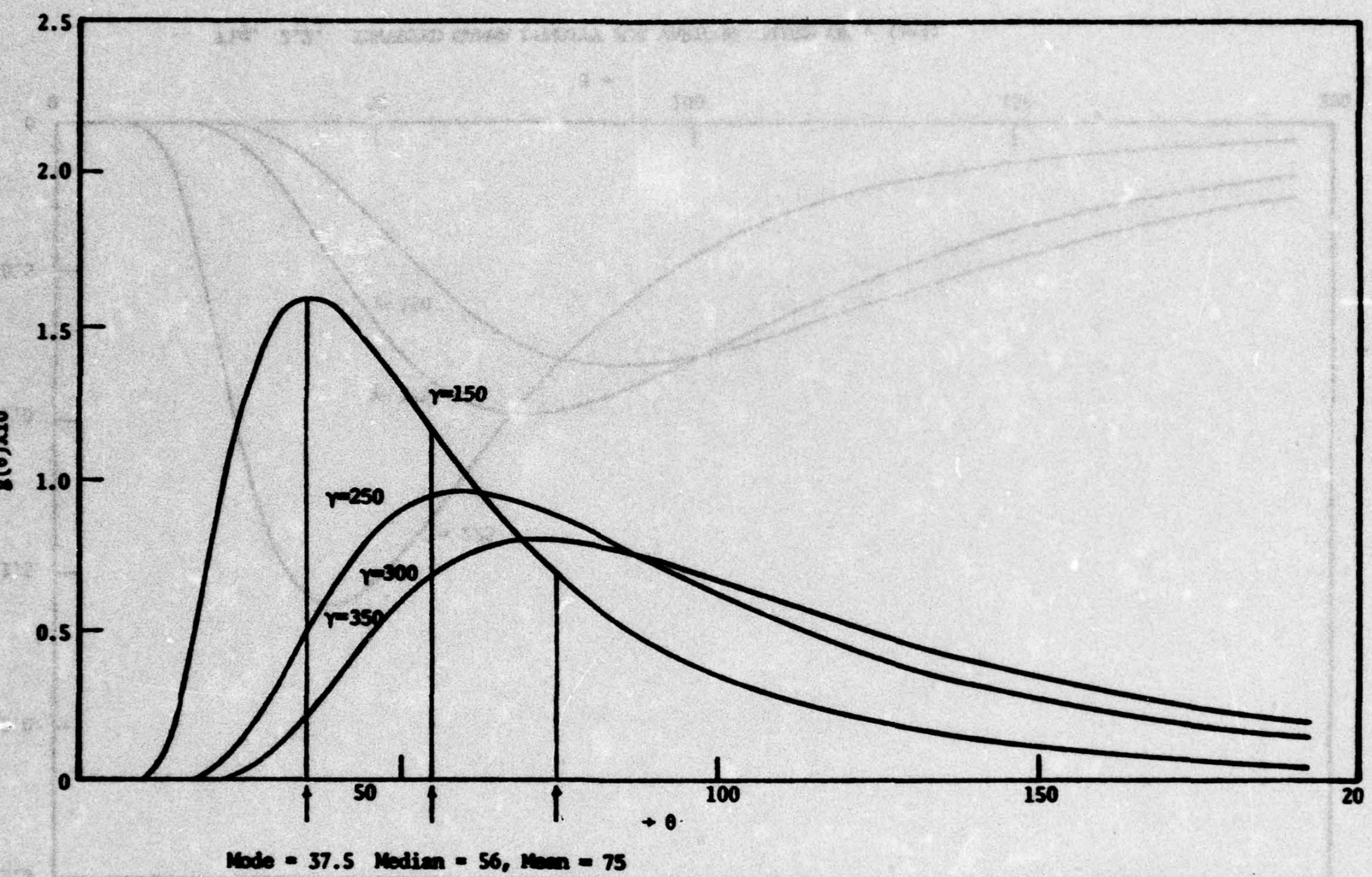

Fig. 2.1. INVERTED GAMMA DENSITY FOR VARIOUS VALUE OF $\gamma(\lambda=3)$

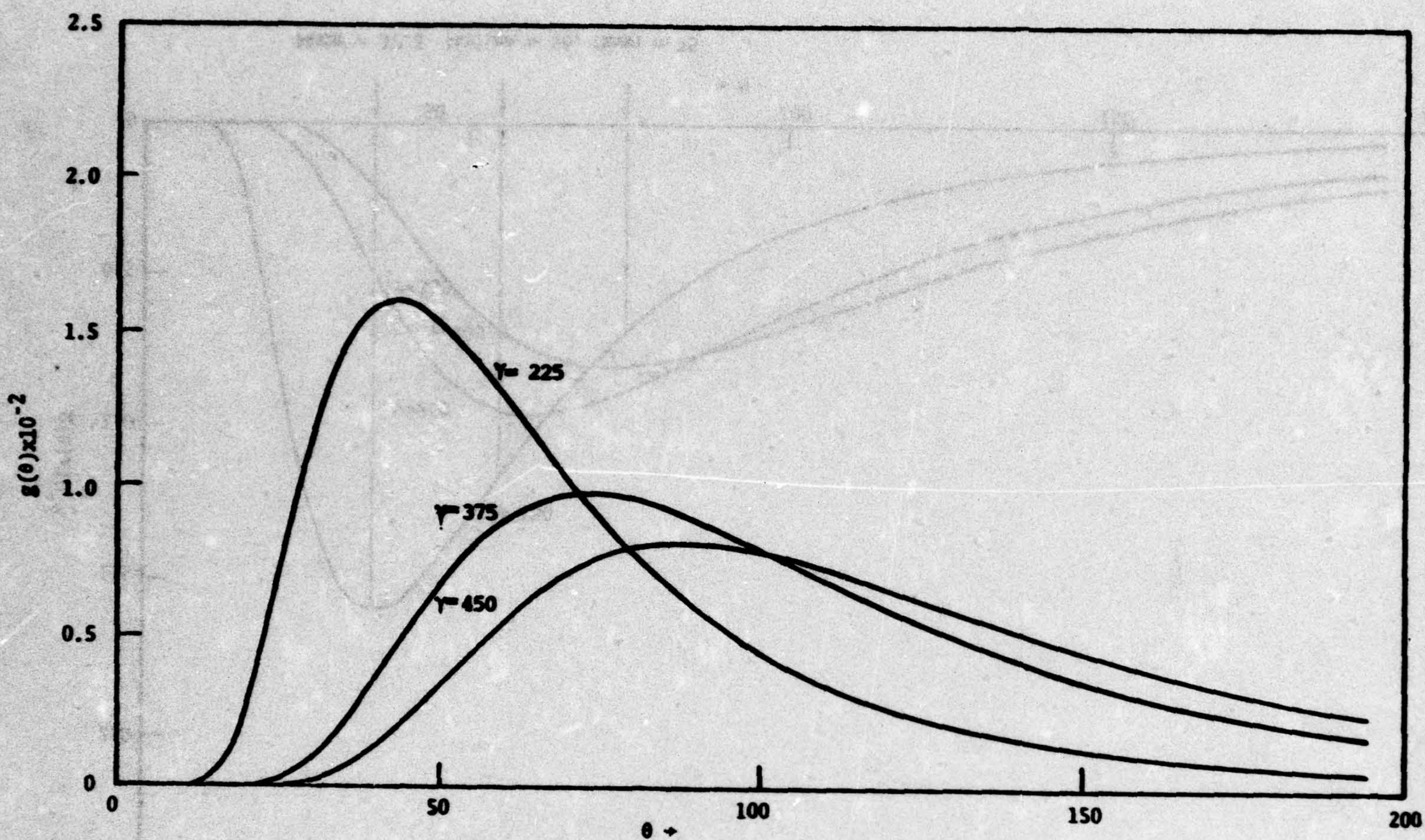

Fig. 2.2. INVERTED GAMMA DENSITY FOR VARIOUS VALUES OF γ (λ=4)

cases are developed in the following subsections. A description of the step by step approach to fitting is given in Appendix A.

Note that if a large number of observations are available on a large number of systems, $g(\hat{\theta})$ can be directly obtained from the observed values of times to failure. Further, $g(\hat{\theta})$ is a consistent estimator of $g(\theta)$ and hence can be used in lieu of $g(\theta)$.

3.2.1. Data From Truncated Plan For a System: (Type II Censoring)

In this case n identical systems are put on a life test of duration T each. The random variable X denotes the number of failures in time T. The times to failure for each system are noted and the resulting data is of the following form:

$$\begin{array}{cccccc}
\theta^1 & \theta^2 & \cdots\cdots & \theta^i & \cdots\cdots & \theta^n \\
t_{11} & t_{21} & & t_{i1} & & t_{n1} \\
t_{12} & t_{22} & & t_{i2} & & t_{n2} \\
\vdots & \vdots & & \vdots & & \vdots \\
\underline{t_{1x_1}} & \underline{t_{2x_2}} & & \underline{t_{ix_i}} & & \underline{t_{nx_n}} \\
\bar{t}_1 & \bar{t}_2 & & \bar{t}_i & & \bar{t}_n
\end{array}$$

Thus, for system i with an unknown MTBF θ^i the x_i failures occur at time intervals $t_{i1}, t_{i2}, \ldots, t_{ix_i}$ and the average time between failures is $\bar{t}_i$.

Since we have assumed that

$$f(t|\theta) = \frac{1}{\theta}\, e^{-t/\theta}, \qquad t \geq 0,\ \theta > 0, \tag{31}$$

it follows that

$$f(x|\theta) = \frac{e^{-T/\theta}(T/\theta)^{x}}{x!}. \tag{32}$$

The marginal distribution of x is

$$f(x) = \int_0^\infty f(x|\theta)\, g(\theta)d\theta$$

$$= \int_0^\infty \frac{e^{-T/\theta}(T/\theta)^x}{x!} \cdot \frac{\gamma^\lambda}{\Gamma(\lambda)}\, \theta^{-(\lambda+1)} e^{-\gamma/\theta} d\theta$$

or

$$f(x) = \frac{\gamma^\lambda}{\Gamma(\lambda)}\, \frac{T^x}{x!} \int_0^\infty \frac{e^{-(T+\gamma)/\theta}}{\theta^{x+\lambda+1}}\, d\theta \tag{33}$$

By letting $Z = (T+\gamma)/\theta$, we get

$$f(x) = \frac{T^x}{x!}\, \frac{\gamma^\lambda}{\Gamma(\lambda)}\, \frac{1}{(T+\lambda)^{\lambda+x}} \int_0^\infty e^{-Z}\, Z^{(\lambda+x-1)} dZ \tag{34}$$

The integral term is $\Gamma(\lambda+x)$, whence

$$f(x) = \frac{\Gamma(\lambda+x)}{\Gamma(\lambda)x!} \left(\frac{T}{T+\gamma}\right)^x \left(\frac{\gamma}{T+\gamma}\right)^\lambda, \qquad x = 0,1,2\ldots \tag{35}$$

i.e. x has a negative binomial density with parameters γ, λ. Therefore estimates $\hat{\gamma}$, $\hat{\lambda}$ of γ and λ can be obtained by fitting a negative binomial distribution to the observations $x_1, x_2, \ldots, x_n$. A computer program for obtaining such a fit is described in Appendix A.

3.2.2. Data From Censored Plan For a System: (Type I Censoring)

In this case n identical systems are put on a life test and testing is continued until each system experiences r failures. The resultant data is of the following form:

$$
\begin{array}{ccccccc}
\theta^1 & \theta^2 & \ldots\ldots & \theta^i & \ldots\ldots & \hat{\theta}^n \\
t_{11} & t_{21} & & t_{i1} & & t_{n1} \\
t_{12} & t_{22} & & t_{i2} & & t_{n2} \\
\cdot & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & & \cdot \\
\cdot & \cdot & & \cdot & & \cdot \\
t_{1r} & t_{2r} & & t_{ir} & & t_{nr} \\
\hline
\hat{\theta}^1 = \bar{t}_1 & \hat{\theta}^2 = \bar{t}_2 & & \hat{\theta}^i = \bar{t}_i & & \hat{\theta}^n = \bar{t}_n
\end{array}
$$

Thus, for system i with an unknown MTBF θ^i the r failures occur at time intervals $t_{i1}, t_{i2}, \ldots t_{ir}$ and the average time between failures is $\bar{t}_i$. Let the random variable T denote the total time on test for a system. T takes values $T_i = \sum_{j=1}^{r} t_{ij}$, $i = 1,2,\ldots n$. Since T is a summation of r exponential random variables, it has the following gamma density.

$$
f(T|\theta) = \frac{1}{\Gamma(r)\theta^r} T^{r-1} e^{-T/\theta}, \quad T \geq 0,\ \theta > 0 \tag{36}
$$

Using the prior $g(\theta)$ from Eq. (27), we get

$$
f(T) = \frac{T^{r-1}}{\Gamma(r)} \frac{\gamma^\lambda}{\Gamma(\lambda)} \int_0^\infty \theta^{-(\lambda+r+1)} e^{-(T+\gamma)/\theta}\, d\theta \tag{37}
$$

By letting $Z = (T+\gamma)/\theta$, we get

$$f(T) = \frac{T^{r-1}}{\Gamma(r)} \frac{\gamma^{\lambda}}{\Gamma(\lambda)} \frac{1}{(T+\gamma)^{\lambda+r}} \int_0^{\infty} Z^{\lambda+r-1} e^{-Z} dZ \qquad (38)$$

The integrated term is $\Gamma(\lambda+r)$ and hence

$$f(T) = \frac{\Gamma(\lambda+r)}{\Gamma(\lambda)\Gamma(r)} \frac{T^{r-1}\gamma^{\lambda}}{(T+\gamma)^{\lambda+r}} \qquad (39)$$

Let $y = T/\gamma$, then

$$f(y) = \frac{\Gamma(\lambda+r)}{\Gamma(\lambda)\Gamma(r)} y^{r-1} \quad (1+y)^{-(\lambda+r)}; \; y > 0, \qquad (40)$$

i.e. y has an inverted beta distribution with parameters λ and r or $y \sim Be'(\lambda,r)$. Since $\hat{\theta} = T/r = (\gamma/r)y$, we get $\hat{\theta} \sim (\gamma/r)\ Be'(\lambda,r)$. Thus a rescaled inverted Beta distribution can be fitted to the observed values $\hat{\theta}^1, \hat{\theta}^2, \ldots, \hat{\theta}^n$ to get estimates of γ and λ.

3.2.3. Data From Censored Plan for a Lot (With and Without Replacement):

In this case k components from each of the n lots are put on a life test and testing for each lot is terminated as soon as exactly $r \leq k$ failures are observed. If T denotes the total time on test for a lot then it follows from Epstein (1960) that, both with and without replacement

$2T/\theta \sim \chi^2_{2r}$. In the replacement case $T = kt_r$ and in the non-replacement case $T = \sum_{i=1}^{r} t_i + (k-r)t_r$. In each case $\hat{\theta} = T/r$. If $y = 2T/\theta$, then

$$f(y|\theta) = \frac{1}{\Gamma(r)2^r} y^{r-1} e^{-y/2} ; \quad y > 0, \tag{41}$$

Hence

$$f(T|\theta) = \frac{1}{\Gamma(r)\theta^r} T^{r-1} e^{-T/\theta}; \quad T > 0 \tag{42}$$

which is the same gamma density as given by Eq. (36). It follows that $\hat{\theta} \sim (\gamma/r)\ Be'(\lambda,r)$ distribution or $(\frac{r}{\gamma})\hat{\theta}$ has an inverted Beta distribution with parameters λ and r.

3.2.4. Data From Truncated Plan for a Lot (With Replacement):

In this case K items from each of the n lots are placed on a life test of duration t_o. Let $x_1, x_2, \ldots, x_n$ denote the number of observed failures in lots 1, 2, ...,n respectively. It is clear that in this case the conditional distribution of the number of failures in t_o is:

$$f(x|\theta) = \frac{e^{-Kt_o/\theta}\,(Kt_o/\theta)^x}{x!}, \; x = 0,1,2,\ldots,K. \tag{43}$$

Hence:

$$f(x) = \frac{\Gamma(\lambda+x)}{\Gamma(\lambda)x!}\left(\frac{Kt_o}{Kt_o+\gamma}\right)^x \left(\frac{\gamma}{Kt_o+\gamma}\right)^\lambda; \; \gamma,\lambda > 0, \tag{44}$$

which is negative binomial with parameters γ and λ.

3.2.5. Data From Truncated Plan for a Lot (Without Replacement):

In this case k items from each of the n lots are placed on a life test of duration t_o. Let $R = 1 - \exp(-t_o/\theta)$ denote the reliability. Then $(1-R)$ represents the probability of failure. If r denotes the number of failures then

$$f(r|R) = \binom{n}{r} (1-R)^r (R)^{n-r} \tag{45}$$

Prior for R:

Given $\theta \sim Ga'(\gamma,\lambda)$ and $R = e^{-t/\theta}$,

$$g(R) = \frac{\gamma^\lambda}{\Gamma(\lambda)} \frac{1}{t_o} \theta^{-(\lambda-1)} e^{-(\gamma-t_o)/\theta} \tag{46}$$

Since $e^{-(\gamma-t_o)/\theta} = R^{\frac{\gamma}{t_o}-1}$ and $\theta = -t_o/\ln R$, we get

$$g(R) = \frac{(\gamma/t_o)^\lambda}{\Gamma(\lambda)} (-\ln R)^{\lambda-1} R^{\gamma/t_o-1} \tag{47}$$

If $R = e^{-t_o/\theta} \simeq 1$ then $\ln R = (1-R)$ and if $\alpha = \gamma/t_o > 0$, then $\alpha^\lambda = \Gamma(\alpha+\lambda)/\Gamma(\alpha)$, for small λ.

This gives:

$$g(R) = \frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\lambda)} R^{\alpha-1} (1-R)^{\lambda-1}, \quad 0 < R < 1 \tag{48}$$

The approximate marginal density of r is thus given by:

$$f(r) = \int_0^1 f(r|R)\, g(R)dR = \binom{n}{r} \frac{\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\ \Gamma(\lambda)} \cdot \frac{\Gamma(n+\alpha-r)\ \Gamma(r+\lambda)}{\Gamma(n+\alpha+r)}, \tag{49}$$

which is the beta-binomial distribution.

4. FITTING COMPLETE AND TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION

The purpose of this section is to illustrate the estimation of prior parameters and their variance-covariance matrix for the case described in Section 3.2.1. In this case, the data consists of field failures on different types of systems. For each system, the observed failure frequencies n_x, corresponding to the observed number of failures x in a fixed time T, are available. For this situation it has been shown in Section 3.2.1. that if:

$$g(\theta) = \frac{\gamma^{\lambda}}{\Gamma(\lambda)} \theta^{-(\lambda+1)} e^{-\gamma/\theta} \quad \theta,\gamma,\lambda \geq 0 \tag{50}$$

and

$$f(t|\theta) = \frac{1}{\theta} e^{-t/\theta} \qquad t > 0$$

or

$$f(x|\theta) = \frac{e^{-T/\theta}(T/\theta)^x}{x!} \qquad x = 0,1,2,..., \tag{51}$$

then

$$f(x) = \frac{(\lambda+x-1)!}{x!(\lambda-1)!} \left(\frac{T}{T+\gamma}\right)^x \left(\frac{\gamma}{T+\gamma}\right)^{\lambda}; \; x = 0,1,2,... \tag{52}$$

Furthermore, as mentioned earlier, because of the identifiability of the prior, if X has a negative binomial distribution with parameters λ and γ given by Eq. (52) and the conditional failure density is exponential then $g(\theta)$ is an inverted gamma distribution with parameters γ and λ as given by Eq. (27).

The parameters (γ,λ) can be estimated by fitting a NBD(γ,λ) to the observed data (x,n_x). In this case, when the failure time is exponential, the adequacy of the negative binomial fit, which is a compound Poisson distribution, ensures the adequacy of the inverted gamma prior (See Teicher, 1960, p. 71).

Sometimes frequency of zeros is not recorded. In that situation, X follows a truncated negative binomial distribution given by:

$$f(x) = \frac{(\lambda+x-1)!}{x!\,(\lambda-1)!\,\{1-(\frac{\gamma}{T+\gamma})^{\lambda}\}}(\frac{T}{T+\gamma})^{x}\,(\frac{\gamma}{T+\gamma})^{\lambda};\ x = 1,2,... \tag{53}$$

We now consider parameter estimation for the complete and the truncated negative binomial distributions.

4.1. Complete Negative Binomial Distribution

In this case X has a NBD(γ,λ) distribution given by

$$f(x) = \frac{(\lambda+x-1)!}{x!(\lambda-1)!}\,(\frac{T}{T+\gamma})^{x}\,(\frac{\gamma}{T+\gamma})^{\lambda}, \qquad x = 0,1,2,...$$

with mean and variance given by

$$E(x) = \mu = \frac{\lambda T}{\gamma}, \tag{54}$$

and

$$Var(x) = \frac{\lambda T(T+\gamma)}{\gamma^2} \tag{55}$$

There is some advantage in estimation if we write $f(x)$ in terms of (μ,λ). Then we get

$$f(x) = \frac{(\lambda+x-1)!}{x!\,(\lambda-1)!}\left(\frac{\mu}{\mu+\lambda}\right)^x \left(\frac{\lambda}{\mu+\lambda}\right)^\lambda; \qquad x = 0,1,2,\ldots \quad (56)$$

We shall consider estimation for both pairs of parameters (γ,λ) and (μ,λ).

4.1.1. Moment Estimates:

Equating the sample mean $\bar{x}$ and sample variance S^2 to the population mean and variance given by Eqs. (54) and (55), we get

$$\mu' = \bar{x}$$

$$\gamma' = \frac{\bar{x}T}{S^2-\bar{x}} \qquad (57)$$

and

$$\lambda' = \frac{\bar{x}^2}{S^2-\bar{x}},$$

where T is the test time, and μ', γ', λ' represent the calculated values of μ, γ and λ, respectively. $\bar{x}$ as an estimate of μ is fully efficient. It should be noted that if $S^2 < \bar{x}$, the negative binomial may not be appropriate. (Also see discussion on page 60.) It is shown by Anscomb (1950) that the efficiency of the moment solution is at least 90% under the following conditions:

(a) μ is small, $\frac{\lambda}{\mu} > 6$ or $\gamma > 6T$

(b) μ is large, $\lambda > 13$

(c) μ is medium, $(\lambda+\mu)\ (\lambda+2)/\mu \geq 15$ or $(T+\gamma)\ (\lambda+2)/T \geq 15$

4.1.2. Maximum Likelihood Estimates:

Estimation of γ, λ:

If x takes values 0,1,2...K with the correspondong observed frequencies $n_0, n_1, n_2 \ldots n_K$ the log likelihood can be written as

$$\ln L = \sum_{x=0}^{K} n_x . \ln f(x) \tag{58}$$

or

$$\ln L = \sum_{x=0}^{K} n_x \{(\ln(\lambda+x-1)! - \ln(x)! - \ln(\lambda-1)!$$

$$+ \lambda \ln\left(\frac{\gamma}{T+\gamma}\right) + x \ln\left(\frac{T}{T+\gamma}\right)\} \tag{59}$$

Let

$$n = \sum_{x=0}^{K} n_x \tag{60}$$

and

$$n\bar{x} = \sum_{x=0}^{K} x\, n_x$$

Then

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n(\lambda T - \gamma \bar{x})}{(T+\gamma)\gamma} \tag{61}$$

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{x=1}^{K} \{ n_x \sum_{j=1}^{x} \frac{1}{(\lambda+j-1)} \} - n \ln\left(1+\frac{T}{\gamma}\right)$$

or (62)

$$\frac{\partial \ln L}{\partial \lambda} = \sum_{j=1}^{K} \{ \frac{1}{(\lambda+j-1)} \cdot \sum_{x=j}^{K} n_x \} - n \ln\left(1+\frac{T}{\gamma}\right)$$

The maximum likelihood estimates are obtained by solving

$$\frac{\partial \ln L}{\partial \gamma} = 0,$$

and $$\frac{\partial \ln L}{\partial \lambda} = 0 \qquad (63)$$

The two simultaneous equations are complicated and the solutions γ and λ can not be obtained directly. A general method is to assume a trial solution (γ', λ'), which may be taken as the moment estimates, and derive linear equations for small additive solutions. Such a method of scoring is now described (Rao 1965).

Expanding $\partial \ln L/\partial\gamma$, $\partial \ln L/\partial\lambda$ about (γ', λ') and retaining only the first power of $\delta\gamma = \gamma - \gamma'$, $\delta\lambda = \lambda - \lambda'$ leads to

$$\frac{\partial \ln L}{\partial \gamma} \approx \left\{\frac{\partial \ln L}{\partial \gamma} + \delta\gamma \frac{\partial^2 \ln L}{\partial \gamma^2} + \gamma\lambda \frac{\partial^2 \ln L}{\partial \gamma \partial \lambda}\right\}\Bigg|_{(\gamma=\gamma', \lambda=\lambda')} = 0$$

$$\frac{\partial \ln L}{\partial \lambda} \approx \left\{\frac{\partial \ln L}{\partial \lambda} + \delta\lambda \frac{\partial^2 \ln L}{\partial \lambda^2} + \delta\lambda \frac{\partial^2 \ln L}{\partial \delta \partial \lambda}\right\}\Bigg|_{(\gamma=\gamma', \lambda=\lambda')} = 0 \qquad (64)$$

The incremental corrections are obtained as:

$$\begin{bmatrix} \delta\gamma \\ \delta\lambda \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial^2 \ln L}{\partial \gamma^2} & \dfrac{\partial^2 \ln L}{\partial \gamma \partial \lambda} \\ \dfrac{\partial^2 \ln L}{\partial \gamma \partial \lambda} & \dfrac{\partial^2 \ln L}{\partial \lambda^2} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial \ln L}{\partial \gamma} \\ \dfrac{\partial \ln L}{\partial \lambda} \end{bmatrix}\Bigg|_{(\gamma=\gamma', \lambda=\lambda')} \qquad (65)$$

The parameter values after the first iteration are $\gamma'' = \gamma' + \delta\gamma$ and $\lambda'' = \lambda' + \delta\lambda$. The iterations are continued until convergence occurs.

The second partial derivatives in Eq. (65) are given by

$$\frac{\partial^2 \ln L}{\partial \gamma^2} = \frac{n\bar{x}\gamma^2 - n\lambda T(T+2\gamma)}{\gamma^2 (T+\gamma)^2}, \tag{66}$$

$$\frac{\partial^2 \ln L}{\partial \gamma \partial \lambda} = \frac{nT}{\gamma(T+\gamma)}, \tag{67}$$

and
$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -\sum_{x=1}^{K} \{n_x \sum_{j=1}^{x} \frac{1}{(\lambda+j-1)^2}\} = -\sum_{j=1}^{K} \{\frac{1}{(\lambda+j-1)^2} \sum_{x=j}^{K} n_x\} \tag{68}$$

Variance-Covariance matrix for $\hat{\gamma}, \hat{\lambda}$

The variance-covariance matrix for $\hat{\gamma}, \hat{\lambda}$ is given by

$$\begin{bmatrix} \mathrm{Var}(\hat{\gamma}) & \mathrm{cov}(\hat{\gamma},\hat{\lambda}) \\ \mathrm{Cov}(\hat{\gamma},\hat{\lambda}) & \mathrm{var}(\hat{\lambda}) \end{bmatrix} = -\begin{bmatrix} E\frac{\partial^2 \ln L}{\partial \gamma^2} & E\frac{\partial^2 \ln L}{\partial \gamma \partial \lambda} \\ E\frac{\partial^2 \ln L}{\partial \gamma \partial \lambda} & E\frac{\partial^2 \ln L}{\partial \lambda^2} \end{bmatrix}^{-1} \tag{69}$$

where, from Eqs. (66), (67) and (68)

$$E\frac{\partial^2 \ln L}{\partial \gamma^2} = -\frac{n\lambda T}{\gamma^2 (T+\gamma)} \tag{70}$$

$$E\frac{\partial^2 \ln L}{\partial \gamma \partial \lambda} = \frac{nT}{\gamma(T+\gamma)} \tag{71}$$

and $$E\frac{\partial^2 \ln L}{\partial\lambda^2} = -n\left\{\frac{T}{\lambda(T+\gamma)} + \frac{T^2}{2\lambda(\lambda+1)\ (T+\gamma)^2}\cdot\right.$$

$$\left.\left[1+2\sum_{j=2}^{\infty}\frac{\left(\frac{j}{j+1}\right)\left(\frac{T}{T+\gamma}\right)^{j-1}}{\binom{j+\lambda}{j-1}}\right]\right\} \tag{72}$$

Substituting in Eq. (69)

$$\mathrm{var}(\hat{\lambda}) = \frac{2\lambda(\lambda+1)}{n\left(\frac{T}{T+\gamma}\right)^2\left[1+2\sum_{j=2}^{\infty}\frac{\left(\frac{j}{j+1}\right)\left(\frac{T}{T+\gamma}\right)^{j-1}}{\binom{j+\lambda}{j-1}}\right]} \tag{73}$$

$$\mathrm{var}(\hat{\gamma}) = \frac{\gamma^2}{n\lambda\left(\frac{T}{T+\gamma}\right)} + \frac{2\gamma^2(\lambda+1)}{n\lambda\left(\frac{T}{T+\gamma}\right)^2\left[1+2\sum_{j=2}^{\infty}\frac{\left(\frac{j}{j+1}\right)\left(\frac{T}{T+\gamma}\right)^{j-1}}{\binom{j+\gamma}{j-1}}\right]} \tag{74}$$

$$\mathrm{cov}(\hat{\gamma},\hat{\lambda}) = \frac{2\gamma(\lambda+1)}{n\left(\frac{T}{T+\gamma}\right)^2\left[1+2\sum_{j=2}^{\infty}\frac{\left(\frac{j}{j+1}\right)\left(\frac{T}{T+\gamma}\right)^{j-1}}{\binom{j+\gamma}{j-1}}\right]} \tag{75}$$

The correlation coefficient between $\hat{\gamma}$ and $\hat{\lambda}$ is

$$\rho(\hat{\gamma},\hat{\lambda}) = \frac{\mathrm{cov}(\hat{\gamma},\hat{\lambda})}{\mathrm{var}(\hat{\gamma})\,\mathrm{var}(\hat{\lambda})} = \left\{\frac{2(\lambda+1)}{2(\lambda+1)+\left(\frac{T}{T+\gamma}\right)\left[1+2\sum_{j=2}^{\infty}\frac{\left(\frac{j}{j+1}\right)\left(\frac{T}{T+\gamma}\right)^{j-1}}{\binom{j+\lambda}{j-1}}\right]}\right\}^{1/2} \tag{76}$$

In estimating the variance-covariance matrix, γ and λ are replaced by $\hat{\gamma}$ and $\hat{\lambda}$.

Estimation of μ, λ:

Following Fisher (1941, 1953), Haldane (1941), and Anscombe (1950), $\hat{\mu}$ and $\hat{\lambda}$ may be obtained as follows. The log likelihood can be written as

$$\ln L = \sum_{x=0}^{K} n_x \{\ln(\lambda+x-1)! - \ln(\lambda-1)! + x\ln(\frac{\mu}{\mu+\lambda}) + \lambda\ln(\frac{\lambda}{\mu+\lambda}) - \ln(x)!\} \quad (77)$$

Hence

$$\frac{\partial \ln L}{\partial \mu} = \sum_{x=0}^{K} n_x \left\{\frac{x}{\mu(\mu+\lambda)} - \frac{\lambda}{(\mu+\lambda)}\right\} \quad (78)$$

Equating $\partial \ln L/\partial \mu = 0$ gives $\hat{\mu} = \bar{x}$ (79)

Differentiating lnL with respect to λ and substituting $\mu = \bar{x}$, the maximum likelihood equation is

$$\frac{\partial \ln L}{\partial \hat{\lambda}} = \sum_{x=0}^{K} \{n_x \sum_{j=1}^{x} \frac{1}{(\hat{\lambda}+j-1)}\} + n\ln(\frac{\hat{\lambda}}{\bar{x}+\hat{\lambda}}) = 0 \quad (80)$$

This equation can be iteratively solved to obtain $\hat{\lambda}$.

Variance-Covariance Matrix for $(\hat{\mu}, \hat{\lambda})$:

$$\mathrm{var}(\hat{\mu}) = \mathrm{var}(\bar{x}) = (\mu+\mu^2/\lambda)/n \quad (81)$$

Since

$$E\frac{\partial^2 \ln L}{\partial\mu\partial\lambda} = E\left[\sum_{x=0}^{K} n_x \left\{\frac{\mu}{(\mu+\lambda)^2}\left(\frac{x-\mu}{\mu}\right)\right\}\right] = 0, \tag{82}$$

$$\mathrm{cov}(\hat{\mu},\hat{\lambda}) = 0, \tag{83}$$

and as before

$$\mathrm{var}(\hat{\lambda}) = \frac{2\lambda(\lambda+1)}{n\left(\frac{T}{T+\gamma}\right)^2\left[1 + 2\sum_{j=2}^{\infty}\frac{\left(\frac{j}{j+1}\right)\left(\frac{T}{T+\gamma}\right)^{j-1}}{\binom{j+\lambda}{j-1}}\right]} \tag{84}$$

The advantage of considering (μ,λ) as parameters rather than (γ,λ) is that $\hat{\mu}$ and $\hat{\lambda}$ are uncorrelated.

4.2 Truncated Negative Binomial Distribution

When zeros are not recorded, the resulting distribution is a truncated negative binomial given by

$$f(x) = \frac{(\lambda+x-1)!}{x!(\lambda-1)!\left\{1-\left(\frac{\gamma}{T+\gamma}\right)^{\lambda}\right\}}\left(\frac{\gamma}{T+\gamma}\right)^{\lambda}\left(\frac{T}{T+\gamma}\right)^{x},\ x = 1,2,\ldots \tag{85}$$

The mean and variance are given by [Johnson & Kotz (1969)]

$$E(x) = \frac{\lambda T}{\gamma} \{1 - (\frac{\gamma}{T+\gamma})^{\lambda}\}^{-1} \tag{86}$$

$$\mathrm{Var}(x) = \frac{\lambda T(T+\gamma)}{\gamma^2} \{1 - (\frac{\gamma}{T+\gamma})^{\lambda}\}^{-1} \left[1 - (\frac{\lambda T}{T+\gamma}) \{[1-(\frac{\gamma}{T+\gamma})^{\lambda}]^{-1} - 1\}\right] \tag{87}$$

Equating the sample and expected values of mean and variance gives two simultaneous equations for γ and λ from which moment estimates of these parameters can be obtained. The equations do not have explicit solutions and Brass (1958) proposed using the observed proportion of unit values along with $\bar{x}$ and s^2 to obtain γ and λ. If n_1 denotes the frequency of 1, then

$$\gamma' = \frac{\bar{x}T(n-n_1)}{ns^2 - \bar{x}(n-n_1)} \tag{88}$$

and

$$\lambda' = \frac{n\gamma'\bar{x} - n_1(T+\gamma')}{nT} \tag{89}$$

where T is the test time, γ' and λ' represent the calculated values of γ and λ respectively. Brass (1958) also shows that the efficiency of these estimates is about 90% in most cases. A detailed discussion of the efficiency of these estimates is given in Sections 4.2.2 and 4.2.3.

4.2.1 Maximum Likelihood Estimates

The log likelihood is given by

$$\ln L = \sum_{x=1}^{K} n_x \{\ln(\lambda+x-1)! - \ln(x)! - \ln(\lambda-1)! + \lambda \ln(\frac{\gamma}{T+\gamma}) \tag{90}$$

$$+ x \ln(\frac{T}{T+\gamma}) - \ln[1 - (\frac{\gamma}{T+\gamma})^{\lambda}]\}$$

on differentiating the log likelihood function we get

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n\lambda T}{\gamma(T+\gamma)\ \{1-(\frac{\gamma}{T+\gamma})^{\lambda}\}} - \frac{n\bar{x}}{(T+\gamma)}, \tag{91}$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n \ln\ (\frac{\gamma}{T+\gamma})}{1-(\frac{\gamma}{T+\gamma})^{\lambda}} + \sum_{j=1}^{K} \{\frac{1}{(\lambda+j-1)} \sum_{x=j}^{K} n_x\} \tag{92}$$

The maximum likelihood equations obtained by setting $\partial \ln L/\partial\gamma = 0$ and $\partial \ln L/\partial\lambda = 0$ can be iteratively solved using the scoring method given earlier. Parameter estimates due to Brass (1958) may be used as initial estimates.

Variance-Covariance Matrix:

An estimate of the variance-covariance matrix based on the sample values is given by

$$\begin{bmatrix} \mathrm{Var}(\hat{\gamma}) & \mathrm{cov}(\hat{\gamma},\hat{\lambda}) \\ \mathrm{Cov}(\hat{\gamma},\hat{\lambda}) & \mathrm{var}(\hat{\lambda}) \end{bmatrix} = - \begin{bmatrix} E\dfrac{\partial^2 \ln L}{\partial \gamma^2} & E\dfrac{\partial^2 \ln L}{\partial \gamma \partial \lambda} \\ E\dfrac{\partial^2 \ln L}{\partial \gamma \partial \lambda} & E\dfrac{\partial^2 \ln L}{\partial \lambda^2} \end{bmatrix}^{-1}, \tag{93}$$

where

$$\frac{\partial^2 \ln L}{\partial \gamma^2} = - \frac{n\lambda T \;[(T+2\gamma)-(T+2\gamma+\lambda T)(\frac{\gamma}{T+\gamma})^{\lambda}]}{\gamma^2 (T+\gamma)^2 \;\{1-(\frac{\gamma}{T+\gamma})^{\lambda}\}^2} + \frac{n\bar{x}}{(T+\gamma)^2} \tag{94}$$

$$\frac{\partial^2 \ln L}{\partial \gamma \partial \lambda} = \frac{nT\;\{1-(\frac{\gamma}{T+\gamma})^{\lambda}[1-\lambda \ln(\frac{\gamma}{T+\gamma})]\}}{\gamma(T+\gamma)\{\;1-(\frac{\gamma}{T+\gamma})^{\lambda}\}^2} \tag{95}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \frac{n(\frac{\gamma}{T+\gamma})^{\lambda}\{\ln(\frac{\gamma}{T+\gamma})\}^2}{\{1-(\frac{\gamma}{T+\gamma})^{\lambda}\}^2} - \sum_{j=1}^{K}\{\frac{1}{(\lambda+j-1)^2}\sum_{x=j}^{K} n_x\} \tag{96}$$

A program to compute $\hat{\gamma}$, $\hat{\lambda}$ and their variance-covariance matrix is given in Appendix A.

4.2.2. Efficiency of Estimation Methods

In this section we discuss the efficiencies of various estimation procedures used for estimating the parameters γ and λ of the truncated negative binomial distributions. The three methods discussed in section 4.2 are

a) The method of moments.

b) The method of modified moments due to Brass, and

c) The method of maximum likelihood.

The natural question that arises at this point is which method is the most desirable one to use. In order to compare these methods we employ a widely used criterion, the Asymptotic Relative Efficiency (A.R.E) given by

$$\text{A.R.E.} = \frac{1}{\text{Asymptotic generalized variance x Information determinant}} \tag{97}$$

The Asymptotic generalized variance is given by

$$|\text{Var}\ (\lambda_e, \gamma_e)| = \begin{vmatrix} \text{Var}\ (\lambda_e) & \text{Cov}\ (\lambda_e, \gamma_e) \\ \text{Cov}\ (\lambda_e, \gamma_e) & \text{Var}\ (\gamma_e) \end{vmatrix} \tag{98}$$

where λ_e and γ_e represent the estimators of λ and γ respectively obtained by the method under comparison.

The information determinant is given by

$$\left|E\left\{\frac{\partial \log L}{\partial \theta_i}, \frac{\partial \log L}{\partial \theta_j}\right\}\right| \qquad i,j = 1,2 \tag{99}$$

where $\theta_1 = \lambda$, $\theta_2 = \gamma$ and L is the likelihood function. Expressions for the asymptotic generalized variance for various cases are obtained below.

a) Method of Moments

The asymptotic generalized variance for the moment estimators λ_{em} and γ_{em} is given by

$$|\mathrm{Var}\ (\lambda_{em}, \gamma_{em})| = |A_3{}' A_2{}' A_1 A_2 A_3| \tag{100}$$

where A_1, A_2, A_3 are as defined below.

A_1 is the variance covariance matrix of the first two sample moments and is given by

$$A_1 = \begin{pmatrix} \mu_2{}' - \mu_1{}'^2 & \mu_3{}' - \mu_2{}'\mu_1{}' \\ \mu_3{}' - \mu_2{}'\mu_1{}' & \mu_4{}' - \mu_2{}'^2 \end{pmatrix} \tag{101}$$

where $E(X^i) = \mu_i{}'$, $\quad i = 1,2,\ldots.$

A_2 and A_3 are the Jacobians for the transformations

$$(\mu_1{}', \mu_2{}') \rightarrow (\lambda, p), \text{ and} \tag{102}$$

$$(\lambda, p) \rightarrow (\lambda, \gamma) \tag{103}$$

respectively, and $p = T/\gamma$

For the truncated negative binomial distribution.

$$\mu_1{}' = \frac{\lambda(\frac{T}{\gamma})}{1-(\frac{T+\gamma}{\gamma})^{-\lambda}} \tag{104}$$

$$\mu_2{}' = \frac{\lambda(\lambda+1)\ (\frac{T}{\gamma})^2 + \lambda(\frac{T}{\gamma})}{1-(\frac{T+\gamma}{\gamma})^{-\lambda}} \tag{105}$$

Define $A_2 = B^{-1}$. Then B is a 2 x 2 matrix with

$$B(1,1) = \frac{(\frac{T}{\gamma})}{1-(\frac{T+\gamma}{\gamma})^{-\lambda}} - \frac{\lambda(\frac{T}{\gamma})\ (\frac{T+\gamma}{\gamma})^{-\lambda} \cdot \log\ (\frac{T+\gamma}{\gamma})}{\{1 - (\frac{T+\gamma}{\gamma})^{-\lambda}\}^2} \tag{106}$$

$$B(1,2) = \frac{\lambda}{1-(\frac{T+\gamma}{\gamma})^{-\lambda}} - \frac{\lambda^2(\frac{T}{\gamma})\ (\frac{T+\gamma}{\gamma})^{-\lambda-1}}{\{1 - (\frac{T+\gamma}{\gamma})^{-\lambda}\}^2} \tag{107}$$

$$B(2,1) = \frac{(2\lambda+1)\ (\frac{T}{\gamma})^2 + (\frac{T}{\gamma})}{1 - (\frac{T+\gamma}{\gamma})^{-\lambda}} - \frac{\{\lambda(\lambda+1)\ (\frac{T}{\gamma})^2 + \lambda(\frac{T}{\gamma})\}\ (\frac{T+\gamma}{\gamma})^{-\lambda}\ \log(\frac{T+\gamma}{\gamma})}{\{1 - (\frac{T+\gamma}{\gamma})^{-\lambda}\}^2} \tag{108}$$

and

$$B(2,2) = \frac{2\lambda\ (\lambda+1)\ (\frac{T}{\gamma}) + \lambda}{1 - (\frac{T+\gamma}{\gamma})^{-\lambda}} - \frac{\lambda^2\ \{(\lambda+1)\ (\frac{T}{\gamma})^2 + \frac{T}{\gamma}\}\ (\frac{T+\gamma}{\gamma})^{-\lambda-1}}{\{1 - (\frac{T+\gamma}{\gamma})^{-\lambda}\}^2} \tag{109}$$

The Jacobian matrix A_3 is given by

$$A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{\gamma^2}{T} \end{pmatrix} \tag{110}$$

where T is the test time.

The asymptotic generalized variance can be obtained by plugging in the values of A_1, A_2, A_3 into Equations (100) and then A.R.E is obtained from Equations (97) and (99).

b) Method of Modified Moments due to Brass

Asymptotic generalized variance for Brass's modified moment estimators λ_{eb} and γ_{eb} is given by

$$|\mathrm{Var}\ (\lambda_{eb}, \gamma_{eb})| = |A'_5\ A'_4\ A_5| \tag{111}$$

A_4 is the variance-covariance matrix of the first two sample moments and the first sample frequency and is given by

$$A_4 = \begin{pmatrix} \mu_2' - \mu_1'^2 & \mu_3' - \mu_2'\,\mu_1' & (1 - \mu_1')\,p_1 \\ \mu_3' - \mu_1'\,\mu_2' & \mu_4' - \mu_2'^2 & (1 - \mu_2')\,p_1 \\ (1 - \mu_1')\,p_1 & (1 - \mu_2')\,p_1 & p_1\,(1 - p_1) \end{pmatrix} \tag{112}$$

where p_1, the probability of one failure in time T, is

$$p_1 = \frac{\lambda}{\{1 - (\frac{T+\gamma}{\gamma})^{-\lambda}\}} \quad (\frac{\gamma}{T+\gamma})^{\lambda} \quad (\frac{T}{T+\gamma}) \tag{113}$$

A_5 is the Jacobian for the transformation

$$(\mu_1', \mu_2', p_1) \rightarrow (\gamma, \lambda) \tag{114}$$

the elements of the matrix A_5 are given by

$$A_5(1,1) = \frac{T(1-p_1)}{\mu_2' - \mu_1'^2 - \mu_1'(1-p_1)} + \frac{\mu_1' T(1-p_1)\ \{(2\mu_1' + (1-p_1)\}}{\{\mu_2' - \mu_1' - \mu_1'(1-p_1)\}^2}$$

$$A_5(2,1) = \frac{-\mu_1' \ T\ (1-p_1)}{\{\mu_2' - \mu_1'^2 - \mu_1'\ (1-p_1)\}^2}$$

$$A_5(3,1) = \frac{-\mu_1' \ T}{\mu_2' - \mu_1'^2 = \mu_1'\ (1-p_1)} - \frac{\mu_1'^2 \ T\ (1-p_1)}{\{\mu_2' - \mu_1'^2 - \mu_1'\ (1-p_1)\}^2}$$

$$A_5(1,2) = \{A_5(1,1) \cdot (\mu_1' - p_1) + \gamma\}/T$$

$$A_5(2,2) = A_5(2,1) \cdot (\mu_1' - p_1)/T$$

$$A_5(3,2) = \{A_5(3,1) \cdot (\mu_1' - p_1) - (T+\gamma)\}/T$$

c) Method of Maximum Likelihood

The asymptotic generalized variance for the maximum likelihood estimation procedure turns out to be 1/(Information determinant). Therefore, it is clear that the A.R.E. of MLE is always one. In fact A.R.E. is the asymptotic efficiency of the method under consideration compared to the method of maximum likelihood.

4.2.3. A.R.E. Computations and Comparisons of Estimation Methods

The asymptotic relative efficiencies for the moment estimators for $\lambda = 0.1$ to 5.0 and $p = 1.0$ to 10.0 were computed using the expressions given in Section 4.2.2. and are given in Table 1(a). From this Table we note that for constant p, A.R.E. improves as λ gets larger. For constant λ, A.R.E. deteriorates with increasing p. Thus, A.R.E.'s are generally better for large λ and small p values. For the type of data that we have analyzed in Section 5, we find that λ values are small and p values are large. The A.R.E.'s for these cases are about 55%, a rather poor value.

A.R.E.'s were also computed for the same range of values of λ and p for the modified method of Brass and are given in Table 1(b). We see that the A.R.E.'s follow a pattern similar to that of the method of moments. The values, however, are consistently larger for the modified moments method.

It should be noted that Brass's modified moments method, which is based on the first two sample moments and the first sample frequency is computationally much simpler than the method of moments. Also, as seen above, A.R.E.'s increase considerably by using the extra information extracted from the first-sample frequency. Even though the increase in A.R.E. is significant, the values are still around 75% for the range of parameter of interest in Section 5.

The minimum chi-square methods described by Katti and Gurland (1962) for the negative binomial distributions can be easily extended to the case of truncated negative binomial and they will certainly have higher A.R.E.'s than the moment estimators. These estimators also will have a simple form, but in obtaining these estimators, it is necessary to solve non-linear equations. Gurland (1963), and Hinz and Gurland (1968) have described the generalized minimum chi-square techniques for some generalized Poisson distributions. These techniques are based on functions of sample frequencies and sample moments. These procedures lead to weighted least square techniques and the estimates can be obtained as solutions of these weighted least-squares. These methods also can be easily modified for the truncated case.

At this point, it should be pointed out that the generalized minimum chi-square methods will have higher A.R.E. when more information is extracted from the data. By using three or four sample moments and two or three sample frequencies we are almost guaranteed (for most of the two parameter families) to get 100% efficiency. However, these sample functions have large asymptotic variances when higher moments and frequencies are involved and hence, these functions change considerably from sample to sample.

The method of maximum likelihood has some good asymptotic properties and this method works fairly well in cases of small samples. Also, the MLE's always have a better A.R.E than any other generalized minimum chi-square method. Considering all these points, we suggest that it is better to use the MLE whenever it is possible to do so. Otherwise, some generalized minimum chi-square method based on three sample moments and may be two or three sample frequencies can be used.

In the problem that we are considering, it is not difficult to get MLE's as described in Section 4.2.1. and illustrated in Section 5. This ensures high efficiency of the estimators.

TABLE 1 a

ASYMPTOTIC RELATIVE EFFICIENCIES OF THE METHOD OF MOMENTS FOR THE

TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION

$p = T/\gamma$ \ λ	0.1	0.2	0.3	0.4	0.5	0.75	1.0	2.0	3.0	5.0
1.0	.806	.812	.817	.822	.827	.838	.847	.875	.894	.916
2.0	.717	.725	.732	.740	.746	.761	.775	.814	.839	.874
3.0	.662	.671	.680	.688	.696	.714	.729	.775	.805	.850
4.0	.624	.634	.644	.653	.661	.680	.697	.747	.782	.835
5.0	.595	.606	.616	.626	.635	.655	.673	.726	.766	.826
6.0	.572	.584	.594	.604	.614	.635	.653	.710	.753	.819
7.0	.554	.566	.577	.587	.597	.619	.638	.697	.743	.815
8.0	.538	.550	.562	.572	.582	.605	.624	.686	.736	.811
9.0	.525	.537	.549	.560	.570	.593	.613	.677	.730	.810
10.0	.513	.526	.538	.549	.559	.583	.603	.669	.725	.809

TABLE 1 b

ASYMPTOTIC RELATIVE EFFICIENCIES OF THE METHOD OF MODIFIED MOMENTS

FOR THE TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION

$p=T/\gamma$ \ λ	0.1	0.2	0.3	0.4	0.5	0.75	1.0	2.0	3.0	5.0
0.1	.992	.993	.994	.994	.994	.995	.996	.997	.997	.995
0.2	.984	.986	.987	.987	.988	.990	.991	.992	.990	.986
0.5	.962	.963	.965	.966	.967	.969	.970	.971	.968	.965
0.8	.941	.943	.944	.945	.947	.948	.950	.950	.950	.951
1.0	.928	.930	.931	.932	.934	.935	.936	.937	.937	.942
2.0	.875	.877	.878	.879	.880	.881	.882	.886	.889	.893
3.0	.837	.838	.839	.840	.840	.842	.843	.847	.849	.860
4.0	.807	.808	.809	.809	.810	.811	.813	.815	.817	.840
5.0	.783	.784	.785	.785	.786	.787	.788	.789	.792	.829
6.0	.763	.764	.764	.765	.766	.767	.768	.767	.774	.821
7.0	.746	.747	.748	.748	.749	.750	.750	.748	.760	.816
8.0	.732	.732	.733	.733	.734	.735	.736	.733	.749	.812
9.0	.719	.719	.720	.720	.721	.722	.721	.719	.740	.810
10.0	.708	.708	.709	.709	.709	.710	.709	.708	.734	.809

5. FITTING OF PRIOR DISTRIBUTIONS FOR EIGHT DATA SETS

We now obtain estimates of the parameters and their standard errors for the eight sets of data reported by Schafer (1970). The data consists of field failures on eight different types of systems. For each system, the observed frequencies n_x, corresponding to the observed number of failures x in a fixed time T, are available. The eight data sets are reproduced in Tables 1.1 to 1.8.

To further explain the data, consider the observed frequencies n_x corresponding to the observed number of failures x for a search indicator IP-128A (Table 1.6). In this case, a total of $n_0 + 55$ search indicators were placed on a life test of duration 4320 hours and the observed number of failures for each search indicator were recorded. For example, 21 indicators failed once, 8 failed twice etc. The frequency n_0, corresponding to $x = 0$, i.e. for the indicators that did not fail, was not recorded.

Since observations corresponding to $x = 0$ are missing, a truncated negative binomial distribution should be fitted to the observed data sets (x, n_x). Schafer (1970) however fits a complete negative binomial. Table 2 gives the predicted frequencies obtained by fitting a truncated negative binomial distributions to the data in Table 1.6. For comparison purposes the predicted frequencies for a complete negative binomial distribution are also given in Table 2. It should be noted that the complete negative binomial distribution was fitted by assuming the frequency $n_0 = 0$. Table 3

TABLE 1.1*

FIELD FAILURE DATA ON MTI REFLECTOR SM-225 (Data Set 1)

Number of Failures, x	Observed Frequency, n_x
1	13
2	8
3	4
4	4
5	1
6	2
7	1
8	4
11	1
13	1
17	1
19	1
TOTAL NUMBER OF FAILURES:	41

* Schafer (1970)

TABLE 1.2*

FIELD FAILURE DATA ON SEARCH MVPS PP-3132 (Data Set 2)

x	n_x
1	11
2	11
3	10
4	7
5	5
6	4
7	1
8	4
9	7
10	3
13	1
14	3
15	1
16	3
19	1
22	1
44	1

TOTAL NUMBER OF FAILURES 74

* Schafer (1970)

TABLE 1.3*

FIELD FAILURE DATA ON OSCILLOSCOPE OS-126 (Data Set 3)

x	n_x
1	14
2	12
3	13
4	6
5	3
6	2
7	1
8	1
11	2
13	2
17	1
28	1
29	1
TOTAL NUMBER OF FAILURES:	59

* Schafer (1970)

TABLE 1.4*

FIELD FAILURE DATA ON VIDEO AMPLIFIER AM-472 (Data Set 4)

x	n_x
1	18
2	11
3	6
4	6
5	1
6	2
7	3
8	1
9	1
10	1
12	1
TOTAL NUMBER OF FAILURES:	51

* Schafer (1970)

TABLE 1.5*

FIELD FAILURE DATA ON SYNCH POWER SUPPLY SN-315 (Data Set 5)

x	n_x
1	22
2	7
3	1
4	4
5	2
6	1
7	3
TOTAL NUMBER OF FAILURES:	40

* Schafer (1970)

TABLE 1.6*

FIELD FAILURE DATA ON SEARCH INDICATOR IP-128A (Data Set 6)

x	n_x
1	21
2	8
3	8
4	2
5	5
6	4
7	3
8	2
9	1
19	1
TOTAL NUMBER OF FAILURES:	55

* Schafer (1970)

TABLE 1.7*

FIELD FAILURE DATA ON AMPLIFIER PANEL PN-8 (Data Set 7)

x	n_x
1	18
2	10
3	5
4	2
5	3
6	3
7	3
9	4
10	4
11	1
12	3
13	1
15	1
TOTAL NUMBER OF FAILURES:	58

* Schafer (1970)

TABLE 1.8*

FIELD FAILURE DATA ON TRANSMITTER (Data Set 8)

x	n_x
1	16
2	10
3	7
4	5
5	3
6	3
7	1
8	1
9	1
12	1
15	1
33	1

TOTAL NUMBER OF FAILURES: 50

* Schafer (1970)

TABLE 2

OBSERVED AND PREDICTED FREQUENCIES FOR SEARCH INDICATOR IP-128A

Observed number of failures, x	Observed frequency, n_x	Predicted Frequency $\hat{n}_x$ Truncated Negative Binomial	Complete Negative Binomial
0			6.4
1	21	19.4	9.4
2	8	11.0	9.6
3	8	7.1	8.2
4	2	4.8	6.5
5	5	3.4	4.8
6	4	2.4	3.4
7	3	1.8	2.3
8	2	1.3	1.6
9-19	2	3.8	2.8
	55	55	55

TABLE 3

χ^2 GOODNESS OF FIT TEST FOR THE DATA OF

TABLE 1.6

(a) Complete Negative Binomial (mle)

Number of Failures	Observed Frequency (n_x)	Expected Frequency ($\hat{n}_x$)	($n_x-\hat{n}_x$)	$(n_x-\hat{n}_x)^2/\hat{n}_x$
0	0	6.4	-6.4	6.4
1	21	9.4	11.6	14.3
2	8	9.6	-1.6	0.3
3	8	8.2	-0.2	0.0
4	2	6.5	-4.5	3.1
5-6	9	8.2	0.8	0.1
7-19	7	6.7	0.3	0.0
Total:	55	55		24.2

Since $\chi^2_{\text{observed}} = 24.2$ and $\chi^2_{4,0.05} = 9.5$, complete negative binomial does not fit.

(b) Truncated Negative Binomial (mle)

Number of Failures	Observed Frequency (n_x)	Expected Frequency ($\hat{n}_x$)	($n_x-\hat{n}_x$)	$(n_x-\hat{n}_x)^2/\hat{n}_x$
1	21	19.4	1.6	0.13
2	8	11.0	-3.0	0.82
3	8	7.1	0.9	0.11
4-5	7	8.2	-1.2	0.17
6-19	11	9.3	1.7	0.31
Total:	55	55		1.54

Since $\chi^2_{\text{observed}} = 1.54$ and $\chi^2_{2,0.05} = 6.0$, the truncated negative binomial is a good fit.

contains the χ^2 goodness of fit tests for the fitted distributions. It is clear that the complete negative binomial does not fit whereas the truncated negative binomial distribution indicates a good fit to the observed data. Moreover, as indicated earlier, the complete negative binomial is not the correct distribution to fit in this case.

In order to see the behavior of the likelihood functions for the data being analyzed, three dimensional plots of the normalized likelihood functions (normalized by its maximum value) for data sets in Tables 1.2, 1.5, 1.6 and 1.8 are shown in Figures 3.1 to 3.4, respectively. From these plots it can be seen that the likelihood functions are unimodal in the range $\lambda>0$, $\gamma>0$ and thus the method of maximum likelihood employed here will in fact give the global maximum value of the likelihood function and not some other local maximum. Using the method of maximum likelihood, the estimated parameter values, asymptotic standard errors, the observed and predicated frequencies and the χ^2 goodness of fit tests for the eight sets of data of Tables 1.1 to 1.8 are given in Tables 4.1 to 4.8. In each case a truncated negative binomial distribution was fitted and is seen to provide a satisfactory fit to the observed data. The results are summarized in Table 5.1. The MLE of λ for Data Set 5 was found to be negative and we have taken a value 0.00001 for $\hat{\lambda}$ in this case as explained below. Also, from Table 5.1, it is interesting to note that $\hat{\lambda}$ for all eight systems lies between 0 and 1. In view of the standard error of $\hat{\lambda}$, λ is close to zero which implies large variability associated with the mean time to failure.

The field failure data on Synch Power Supply SN-315 of Table 1.5 gives a negative value for the MLE of the parameter λ, which does not belong to its acceptable range. In such a case, a reasonable approach is to take a proper limiting form of a given distribution which gives $\hat{\lambda} = 0$. However, the value $\hat{\lambda} = 0$ is also not acceptable, since it yields a degenerate distribution at zero which being truncated at zero does not exist. But we can always choose $\hat{\lambda}$ to be very small, say $\hat{\lambda} = 0.00001$, and use this value of $\hat{\lambda}$ as an estimate of λ. This approach is also supported by the plot of the normalized likelihood function in Figure 3.2 for this data set.

A similar problem may arise, when we try to find the moment estimators or the modified moment estimators proposed by Brass (1958), which are used as initial estimates for the maximum likelihood estimation. One of the following cases may arise:

(i) $\hat{\lambda}$ and $\hat{\gamma}$ are both positive

(ii) $\hat{\lambda}$ and $\hat{\gamma}$ are both negative

(iii) $\hat{\lambda}$ or $\hat{\gamma}$ is equal to zero

Cases (i), (ii), and (iii) will occur if $\bar{x} < S^2$, $\bar{x} > S^2$ and $\bar{x} = S^2$, respectively. If $\bar{x} > S^2$ or if $\hat{\lambda}$ and $\hat{\gamma}$ are negative, the truncated binomial distribution will give a better fit. But here we are considering the distribution of the number of failures which is a truncated compound Poisson distribution and has a population variance $\geq$ the population mean. In such a case, the truncated Poisson distribution with its parameter λ obtained from

$\bar{x} = \frac{\lambda}{1-e^{-\lambda}}$ will be a good choice.

For the sake of comparison, we also obtained estimates of λ, γ etc. using Brass's method of modified moments. The results for the eight data sets are tabulated in Table 5.2. A comparison of the results in Tables 5.1 and 5.2 shows that the variability of Brass's estimators is much larger than that of the MLE's. Also, as pointed out in Section 4.2.3, the A.R.E. of Brass's estimators for the range of parameters of interest here is about 75%, a rather poor value.

The plots of the fitted inverted gamma densities and cumulative distribution functions are given in Figures 4.1 to 4.8.

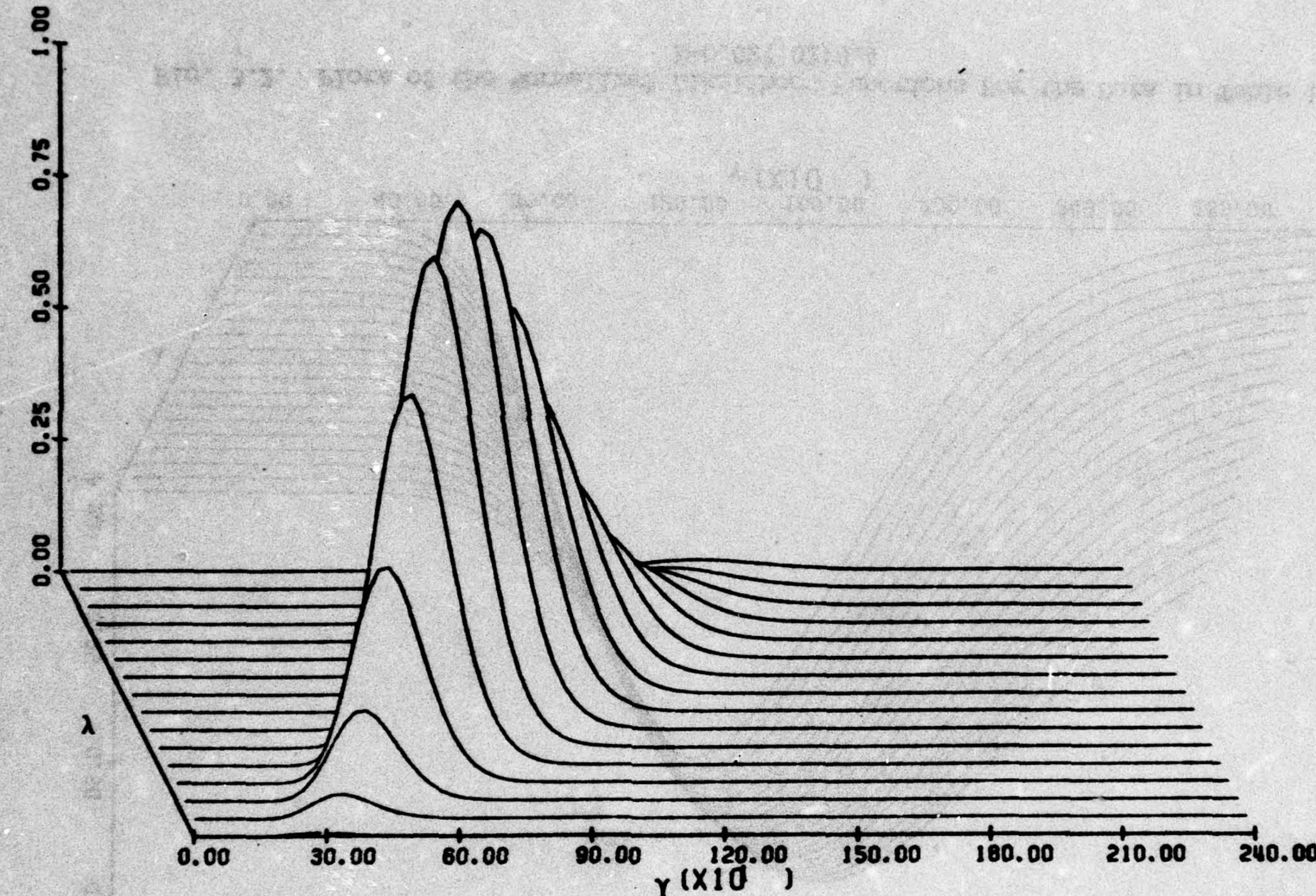

Fig. 3.1. Plots of the Normalized Likelihood Functions For the Data in Table 1.2
λ=0.125(.125)2.0

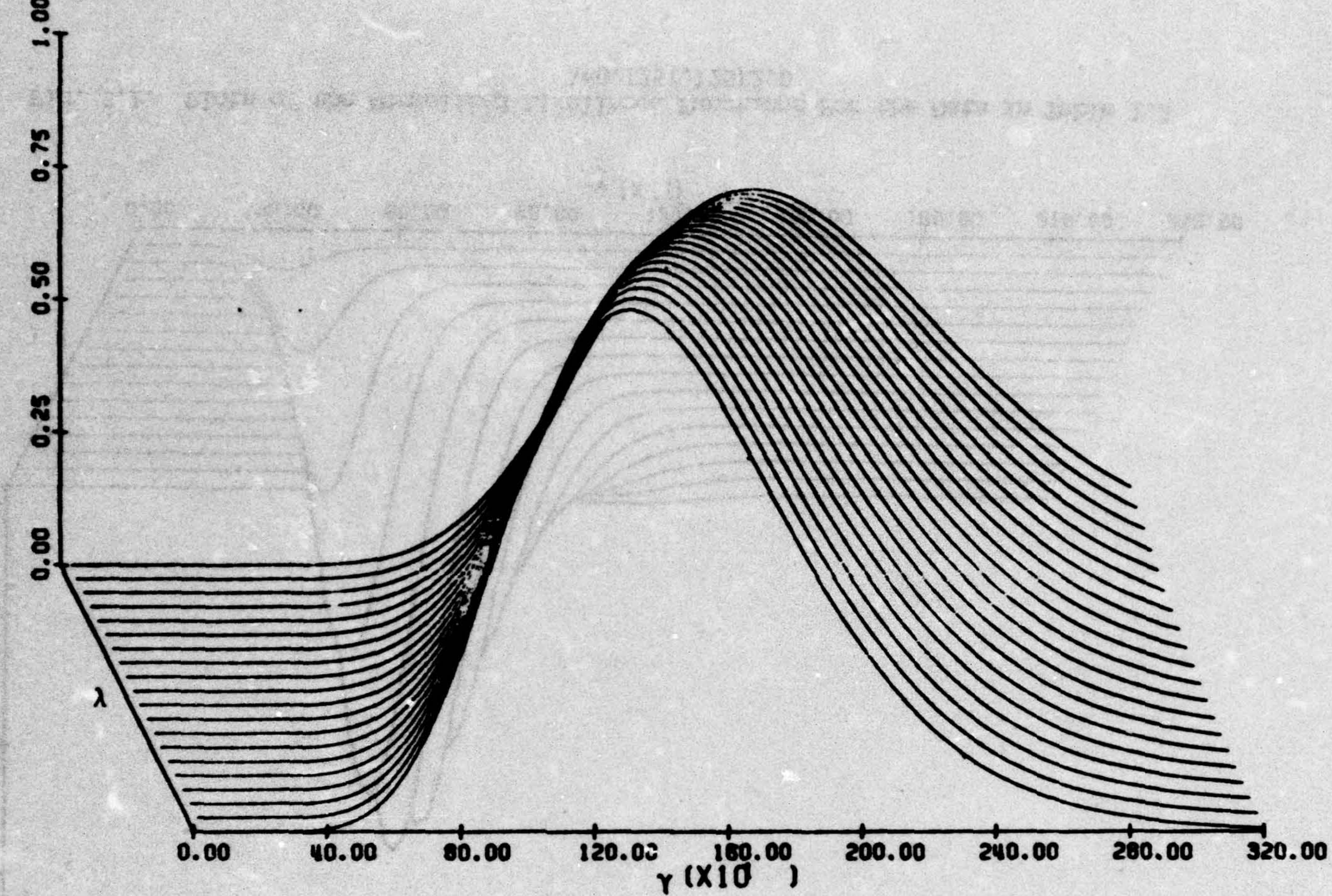

Fig. 3.2. Plots of the Normalized Likelihood Functions For the Data in Table 1.5
λ=0.02(.02)0.4

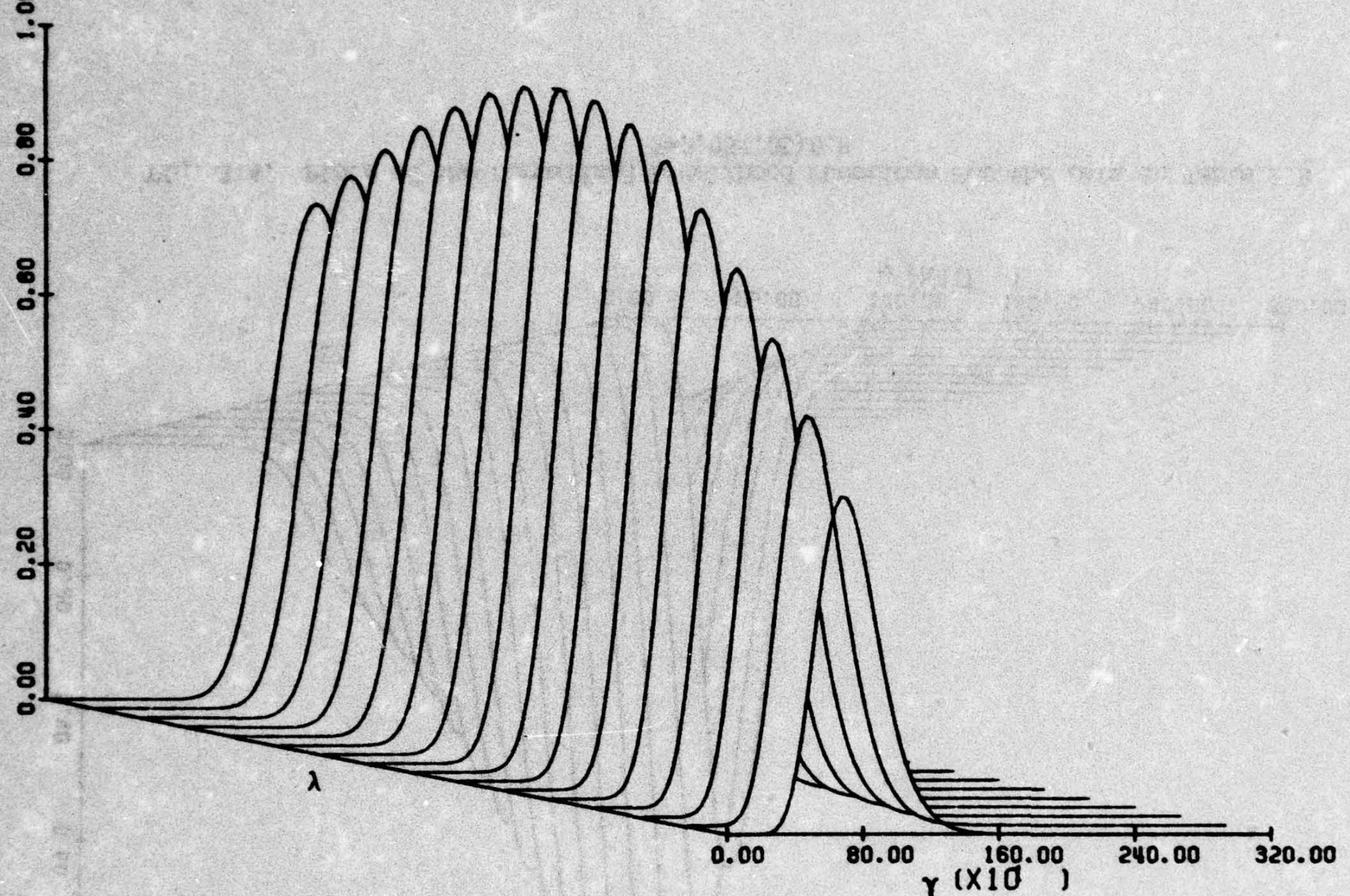

Fig. 3.3. Plots of the Normalized Likelihood Functions For the Data in Table 1.6
$\lambda=0.05(.05)0.8$

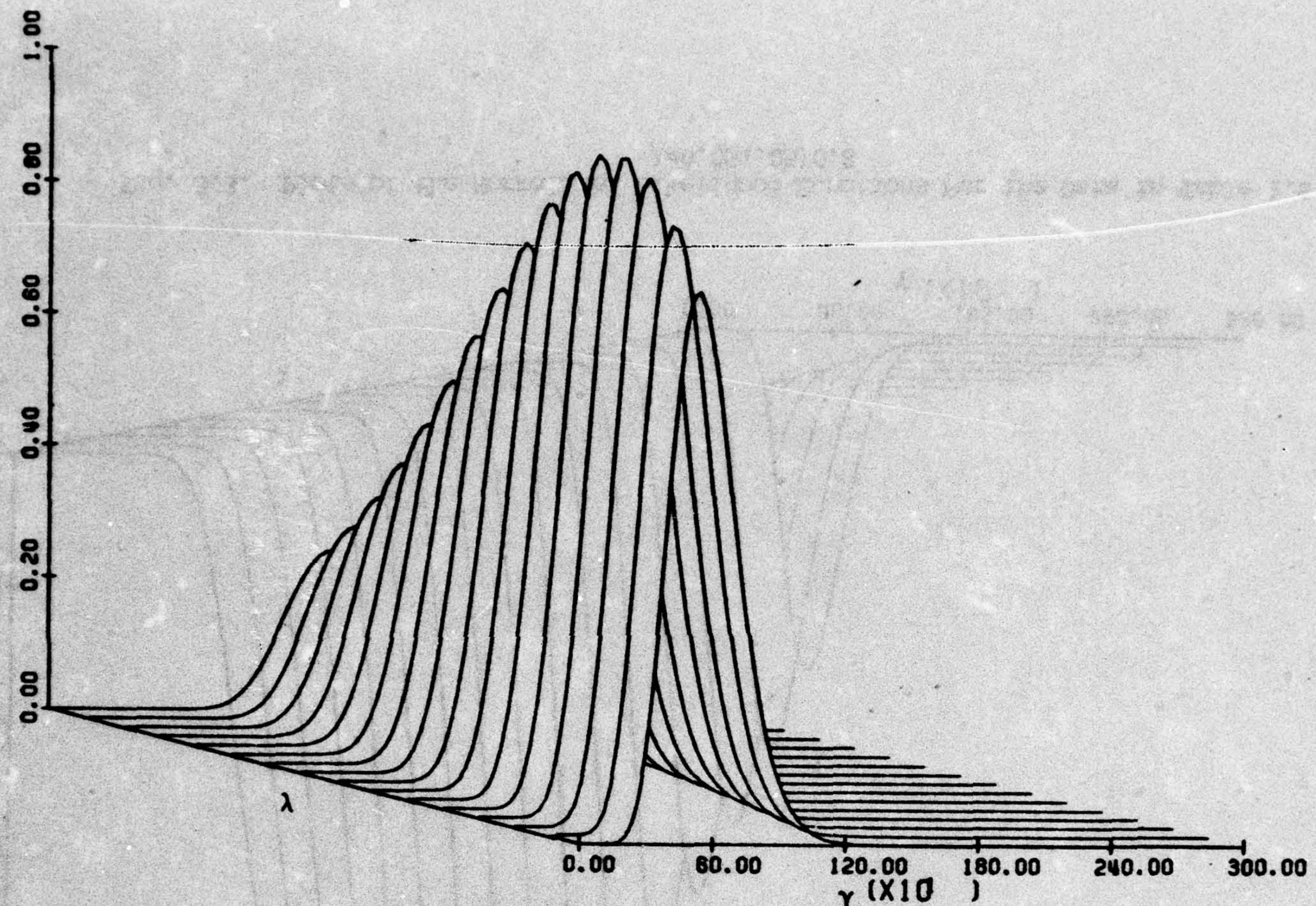

Fig. 3.4. Plots of the Normalized Likelihood Functions For the Data in Table 1.8
$\lambda = 0.05(.05)0.8$

TABLE 4.1

DETERMINATION OF PRIOR: MTI REFLECTOR SM-225

($\hat{\gamma}$= 679.0, $\hat{\lambda}$= 0.2987, $\hat{\sigma}_{\hat{\gamma}}$= 342.4, $\hat{\sigma}_{\hat{\lambda}}$= 0.3333, $\hat{\rho}$= 0.869)

Observed and Predicted Frequencies

x	n_x	$\hat{n}_x$
1	13	13
2	8	7.3
3	4	4.8
4	4	3.4
5	1	2.5
6	2	1.9
7	1	1.5
8	4	1.2
9-19	4	5.4
	41	41

χ^2 Goodness of Fit Test

x	n_x	$\hat{n}_x$	$(n_x-\hat{n}_x)$	$(n_x-\hat{n}_x)^2/\hat{n}_x$
1	13	13	0	0
2	8	7.3	0.7	0.067
3-4	8	8.2	-0.2	0.005
5-7	4	5.9	-1.9	0.612
8-19	8	6.6	1.4	0.297
				0.981

$\chi^2_{\text{observed}} = 0.98$, $\chi^2_{2,0.05} = 5.99$; and $P(\chi^2_2 > \chi^2_{\text{observed}}) = 0.60$

Therefore, truncated negative binomial is a good fit.

TABLE 4.2

DETERMINATION OF PRIOR: SEARCH MVPS PP-3132

($\hat{\gamma}$= 713.7, $\hat{\lambda}$= 0.8646, $\hat{\sigma}_{\hat{\gamma}}$= 210.6, $\hat{\sigma}_{\hat{\lambda}}$= 0.3065, $\hat{\rho}$= 0.892)

Observed and Predicted Frequencies

x	n_x	$\hat{n}_x$	x	n_x	$\hat{n}_x$
1	11	12.4	11	0	2.0
2	11	10.0	12	0	1.7
3	10	8.2	13	1	1.5
4	7	6.8	14	3	1.2
5	5	5.6	15	1	1.1
6	4	4.7	16	3	0.9
7	1	4.0	17-44	3	5.3
8	4	3.4			
9	7	2.8			
10	3	2.4			
	63			74	74

χ^2 Goodness of Fit Test

x	n_x	$\hat{n}_x$	$(n_x-\hat{n}_x)$	$(n_x-\hat{n}_x)^2/\hat{n}_x$
1	11	12.4	-1.4	0.158
2	11	10.0	1.0	0.100
3	10	8.2	1.8	0.395
4	7	6.8	0.2	0.006
5	5	5.6	-0.6	0.064
6-7	5	8.7	-3.7	1.574
8-9	11	6.2	4.8	3.716
10-12	3	6.1	-3.1	1.575
13-44	11	10.0	1.0	0.100
				7.688

$\chi^2_{observed} = 7.688$ and $\chi^2_{6,0.05} = 12.6$, $P(\chi^2_6 > \chi^2_{observed}) = 0.26$

Therefore, truncated negative binomial is a good fit.

TABLE 4.3

DETERMINATION OF PRIOR: OSCILLOSCOPE OS-126

($\hat{\gamma}$= 651.7, $\hat{\lambda}$= 0.3335, $\hat{\sigma}_{\hat{\gamma}}$= 256.1, $\hat{\sigma}_{\hat{\lambda}}$= 0.2616, $\hat{\rho}$= 0.857)

Observed and Predicted Frequencies

x	n_x	$\hat{n}_x$	x	n_x	$\hat{n}_x$
1	14	17.6	11	2	1.0
2	12	10.2	12	0	0.8
3	13	6.9	13	2	0.7
4	6	5.0	14-29	3	3.3
5	3	3.8			
6	2	2.9			
7	1	2.3			
8	1	1.8			
9-13	4	1.5			
14-29	3	1.2			
	59			59	59

χ^2 Goodness of Fit Test

x	n_x	$\hat{n}_x$	$(n_x-\hat{n}_x)$	$(n_x-\hat{n}_x)^2/\hat{n}_x$
1	14	17.6	-3.6	0.736
2	12	10.2	1.8	0.318
3	13	6.9	6.1	5.393
4	6	5.0	1.0	0.200
5-6	5	6.7	-1.7	0.431
7-9	2	5.6	-3.6	2.314
10-29	7	7.0	0.0	0.000
				9.392

$\chi^2_{observed}$ = 9.392 and $\chi^2_{4,0.05}$ = 9.49., and $P(\chi^2_4 > \chi^2_{observed})$ = 0.05. Therefore, truncated negative binomial is a good fit.

TABLE 4.4

DETERMINATION OF PRIOR: VIDEO AMPLIFIER AM-472

($\hat{\gamma}$= 1581.5, $\hat{\lambda}$= 0.6675, $\hat{\sigma}_{\hat{\gamma}}$= 779.7, $\hat{\sigma}_{\hat{\lambda}}$= 0.5277, $\hat{\rho}$= 0.923)

Observed and Predicted Frequencies

x	n_x	$\hat{n}_x$
1	18	17.7
2	11	10.8
3	6	7.0
4	6	4.7
5	1	3.2
6	2	2.2
7	3	1.6
8	1	1.1
9	1	0.8
10-12	2	1.9
	51	51

χ^2 Goodness of Fit Test

x	n_x	$\hat{n}_x$	$(n_x-\hat{n}_x)$	$(n_x-\hat{n}_x)^2/\hat{n}_x$
1	18	17.7	0.3	0.005
2	11	10.8	0.2	0.004
3	6	7.0	-1.0	0.143
4-5	7	7.9	-0.9	0.103
6-12	9	7.6	1.4	0.258
				0.513

$\chi^2_{\text{observed}} = 0.513$ and $\chi^2_{2,0.05} = 5.99$; and $P(\chi^2_2 > \chi^2_{\text{observed}}) = 0.77$.
Therefore, truncated negative binomial is a good fit.

TABLE 4.5

DETERMINATION OF PRIOR: SYNCH POWER SUPPLY SN-315

($\hat{\gamma}$= 1246.0, $\hat{\lambda}$= 0.00001, $\hat{\sigma}_{\hat{\gamma}}$ = 947.4, $\hat{\sigma}_{\hat{\lambda}}$ = 0.4391, $\hat{\rho}$ = 0.908.

Observed and Predicted Frequencies

x	n_x	$\hat{n}_x$
1	22	20.9
2	7	8.0
3	1	4.1
4	4	2.4
5	2	1.5
6	1	1.0
7	3	2.1
	40	40

χ^2 Goodness of Fit Test

x	n_x	$\hat{n}_x$	$(n_x-\hat{n}_x)$	$(n_x-\hat{n}_x)^2/\hat{n}_x$
1	22	20.9	1.1	0.058
2	7	8.0	-1.0	0.125
3-4	5	6.5	-1.5	0.346
5-7	6	4.6	1.4	0.426
				0.955

$\chi^2_{observed}$ = 0.955 and $\chi^2_{1,0.05}$ = 3.84; and $P(\chi^2_1 > \chi^2_{observed})$ = 0.33.

Therefore, truncated negative binomial is a good fit.

TABLE 4.6

DETERMINATION OF PRIOR: SEARCH INDICATOR IP-128A

($\hat{\gamma}$= 1151.8, $\hat{\lambda}$= 0.4407, $\hat{\sigma}_{\hat{\gamma}}$= 542.0, $\hat{\sigma}_{\hat{\lambda}}$= 0.3993, $\hat{\rho}$= 0.905)

Observed and Predicted Frequencies

x	n_x	$\hat{n}_x$
1	21	19.4
2	8	11.0
3	8	7.1
4	2	4.8
5	5	3.4
6	4	2.4
7	3	1.8
8	2	1.3
9-19	2	3.8
	55	55

χ^2 Goodness of Fit Test

x	n_x	$\hat{n}_x$	$(n_x-\hat{n}_x)$	$(n_x-\hat{n}_x)^2/\hat{n}_x$
1	21	19.4	1.6	0.13
2	8	11.0	-3.0	0.82
3	8	7.1	0.9	0.11
4-5	7	8.2	-1.2	0.17
6-19	11	9.3	1.7	0.31
				1.54

$\chi^2_{\text{observed}} = 1.54$ and $\chi^2_{2,0.05} = 5.99$; $P(\chi^2_2 > \chi^2_{\text{observed}}) = 0.46$.

Therefore, truncated negative binomial is a good fit.

TABLE 4.7

DETERMINATION OF PRIOR: AMPLIFIER PANEL PN-8

($\hat{\gamma} = 754.9$, $\hat{\lambda} = 0.4808$, $\hat{\sigma}_{\hat{\gamma}} = 309.9$, $\hat{\sigma}_{\hat{\lambda}} = 0.3286$, $\hat{\rho} = 0.891$)

Observed and Predicted Frequencies

x	n_x	$\hat{n}_x$
1	18	15.8
2	10	10.0
3	5	7.0
4	2	5.2
5	3	4.0
6	3	3.1
7	3	2.4
8	0	1.9
9	4	1.6
10	4	1.2
11	1	1.0
12	3	0.8
13-15	2	4.0
	58	

χ^2 Goodness of Fit Test

x	n_x	$\hat{n}_x$	$(n_x-\hat{n}_x)$	$(n_x-\hat{n}_x)^2/\hat{n}_x$
1	18	15.8	2.2	0.306
2	10	10.0	0.0	0.000
3	5	7.0	-2.0	0.571
4	2	5.2	-3.2	1.969
5-6	6	7.1	-1.1	0.170
7-9	7	5.9	1.1	0.205
10-15	10	7.0	3.0	1.286
				4.507

$\chi^2_{observed} = 4.507$ and $\chi^2_{4,0.05} = 9.49$; and $P(\chi^2_4 > \chi^2_{observed}) = 0.34$.
Therefore, truncated negative binomial is a good fit.

TABLE 4.8

DETERMINATION OF PRIOR: TRANSMITTER

($\hat{\gamma}$= 680.3, $\hat{\lambda}$= 0.2054, $\hat{\sigma}_{\gamma}$= 317.7, $\hat{\sigma}_{\hat{\lambda}}$= 0.2794, $\hat{\rho}$= 0.856)

Observed and Predicted Frequencies

x	n_x	$\hat{n}_x$
1	16	17.5
2	10	9.1
3	7	5.8
4	5	4.0
5	3	2.9
6	3	2.2
7	1	1.7
8	1	1.3
9	1	1.0
10-33	3	4.5
	50	50

χ^2 Goodness of Fit Test

x	n_x	$\hat{n}_x$	$(n_x-\hat{n}_x)$	$(n_x-\hat{n}_x)^2/\hat{n}_x$
1	16	17.5	-1.5	0.129
2	10	9.1	0.9	0.089
3	7	5.8	1.2	0.248
4-5	8	6.9	1.1	0.175
6-8	5	5.2	-0.2	0.008
9-33	4	5.5	-1.5	0.409
				1.055

$\chi^2_{observed}$ = 1.055 and $\chi^2_{3,0.05}$ = 7.81; $P(\chi^2_3 > \chi^2_{observed})$ = 0.79.
Therefore, truncated negative binomial is a good fit.

TABLE 5.1

PARAMETER ESTIMATES AND STANDARD ERRORS USING METHOD

OF MAXIMUM LIKELIHOOD

	Equipment	$\hat{\gamma}$	$\hat{\sigma}_{\hat{\gamma}}$	$\hat{\lambda}$	$\hat{\sigma}_{\hat{\lambda}}$	$\hat{\rho}$
1.	MTI Reflector SM-225	679.0	342.4	0.2987	0.3333	0.869
2.	Search MVPS PP-3132	713.7	210.6	0.8646	0.3065	0.892
3.	Oscilloscope OS-126	651.7	256.1	0.3335	0.2616	0.857
4.	Video Amplifier AM-472	1581.5	779.7	0.6675	0.5277	0.923
5.	Synch Power Supply SN-315	1246.0	947.4	0.00001	0.4391	0.908
6.	Search Indicator IP-128A	1151.8	542.0	0.4407	0.3993	0.905
7.	Amplifier Panel PN-8	754.9	309.9	0.4808	0.3286	0.891
8.	Transmitter	680.3	317.7	0.2054	0.2794	0.856

TABLE 5.2

PARAMETER ESTIMATES AND STANDARD ERRORS USING

BRASS'S MODIFIED METHOD OF MOMENTS

	Equipment	$\hat{\gamma}$	$\hat{\sigma}_{\hat{\gamma}}$	$\hat{\lambda}$	$\hat{\sigma}_{\hat{\lambda}}$	$\hat{\rho}$
1.	MTI Reflector SM-225	775.9	2742.8	0.3838	2.5321	.8943
2.	Search MVPS PP-3132	619.7	1808.9	0.7508	2.6950	.9033
3.	Oscilloscope OS-126	523.4	1861.9	0.2781	2.0944	.8764
4.	Video Amplifier AM-472	1754.0	6529.5	0.7696	4.2833	.9323
5.	Synch Power Supply SN-315	1676.1	7832.9	0.1290	3.2113	.9134
6.	Search Indicator IP-128A	1134.2	4287.4	0.3915	2.9872	.9074
7.	Amplifier Panel	1058.5	3445.2	0.7374	3.3014	.9161
8.	Transmitter	484.8	1942.2	0.0795	1.8580	.8611

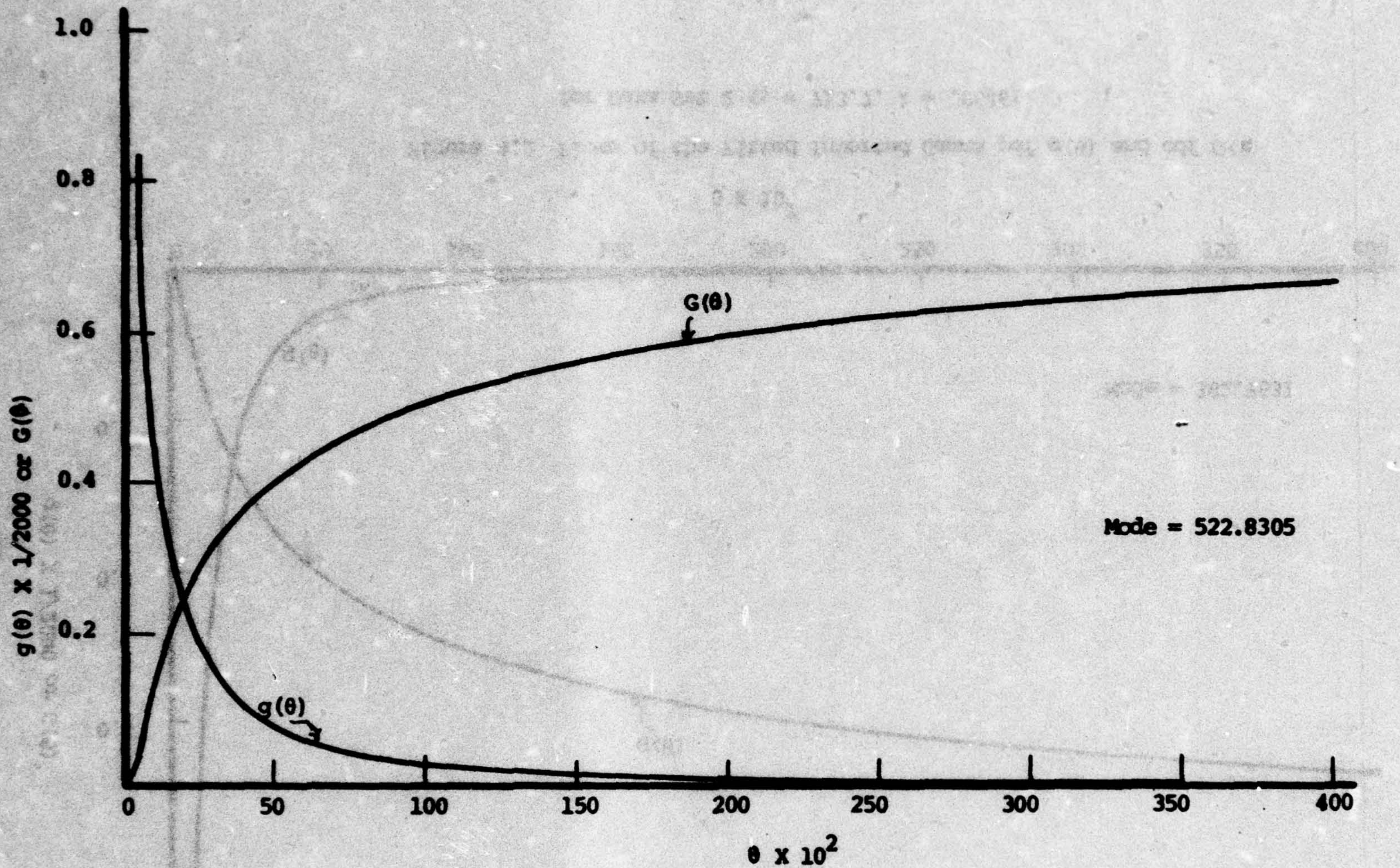

Figure 4.1 Plots of the Fitted Inverted Gamma pdf $g(\theta)$ and cdf $G(\theta)$ for Data Set 1 ($\gamma = 679$, $\lambda = .2987$)

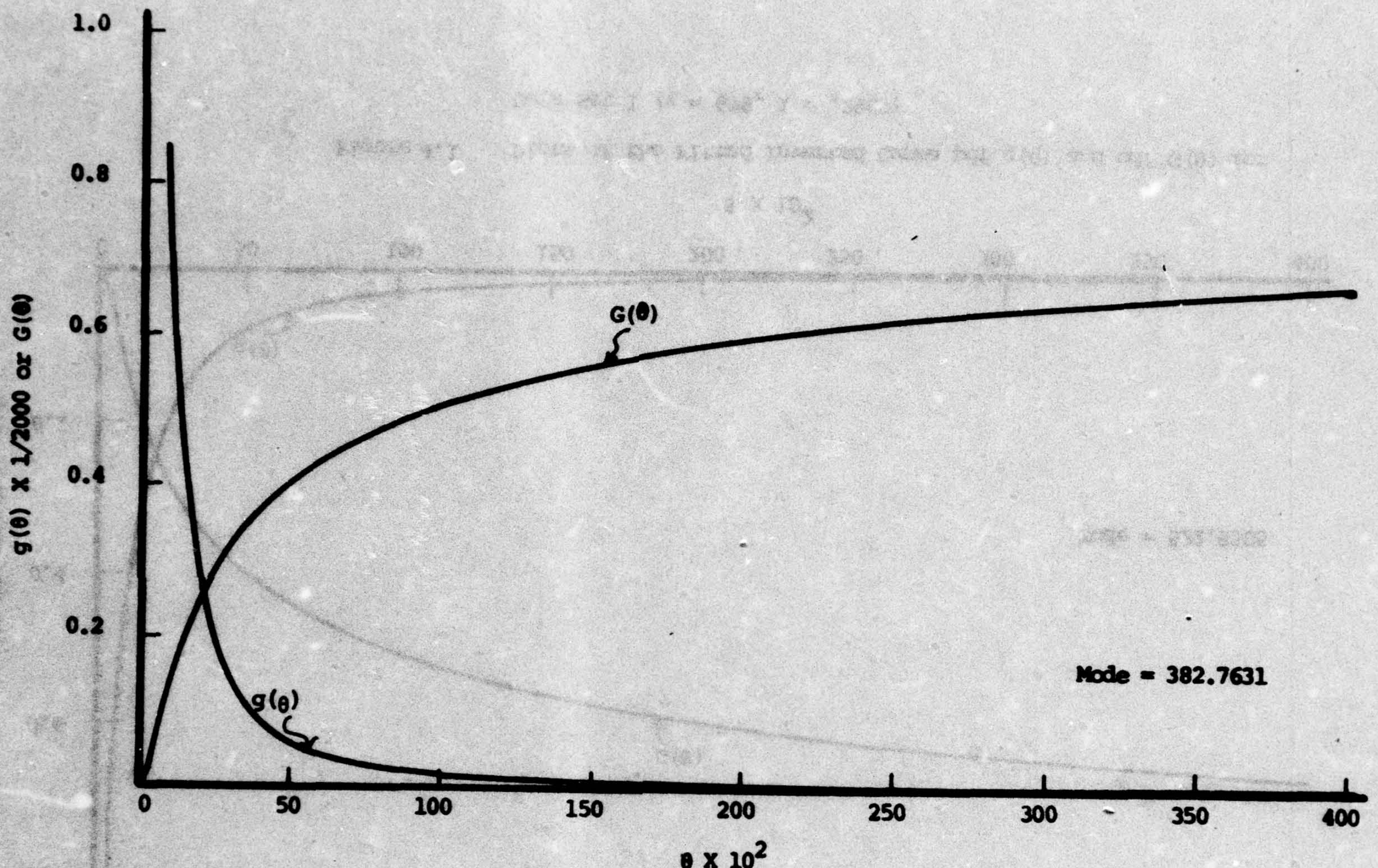

Figure 4.2 Plots of the Fitted Inverted Gamma pdf $g(\theta)$ and cdf $G(\theta)$ for Data Set 2 (γ = 713.7, λ = .8646)

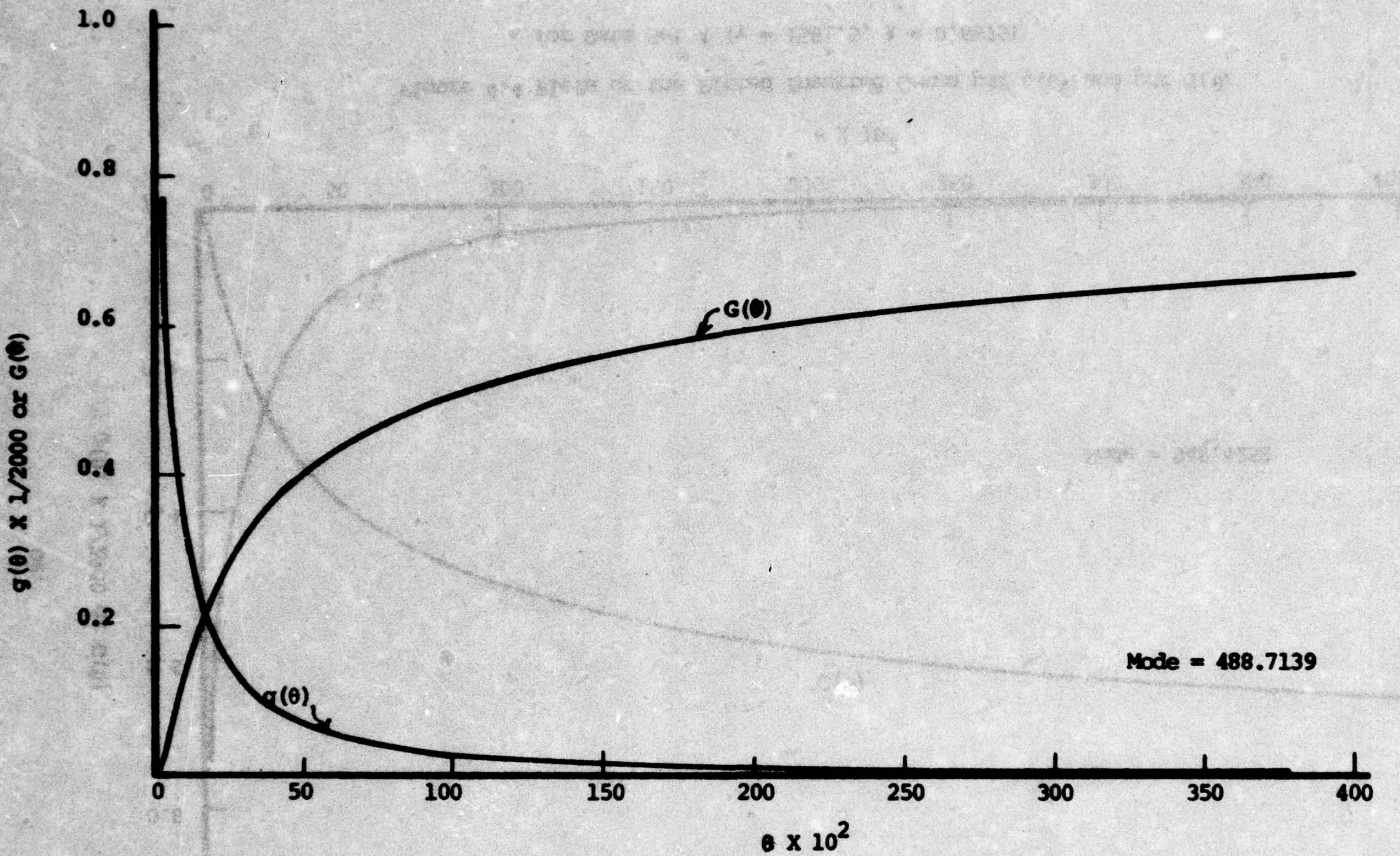

Figure 4.3 Plots of the Fitted Inverted Gamma pdf $g(\theta)$ and cdf $G(\theta)$ for Data Set 3 ($\gamma = 651.7$, $\lambda = 0.3335$)

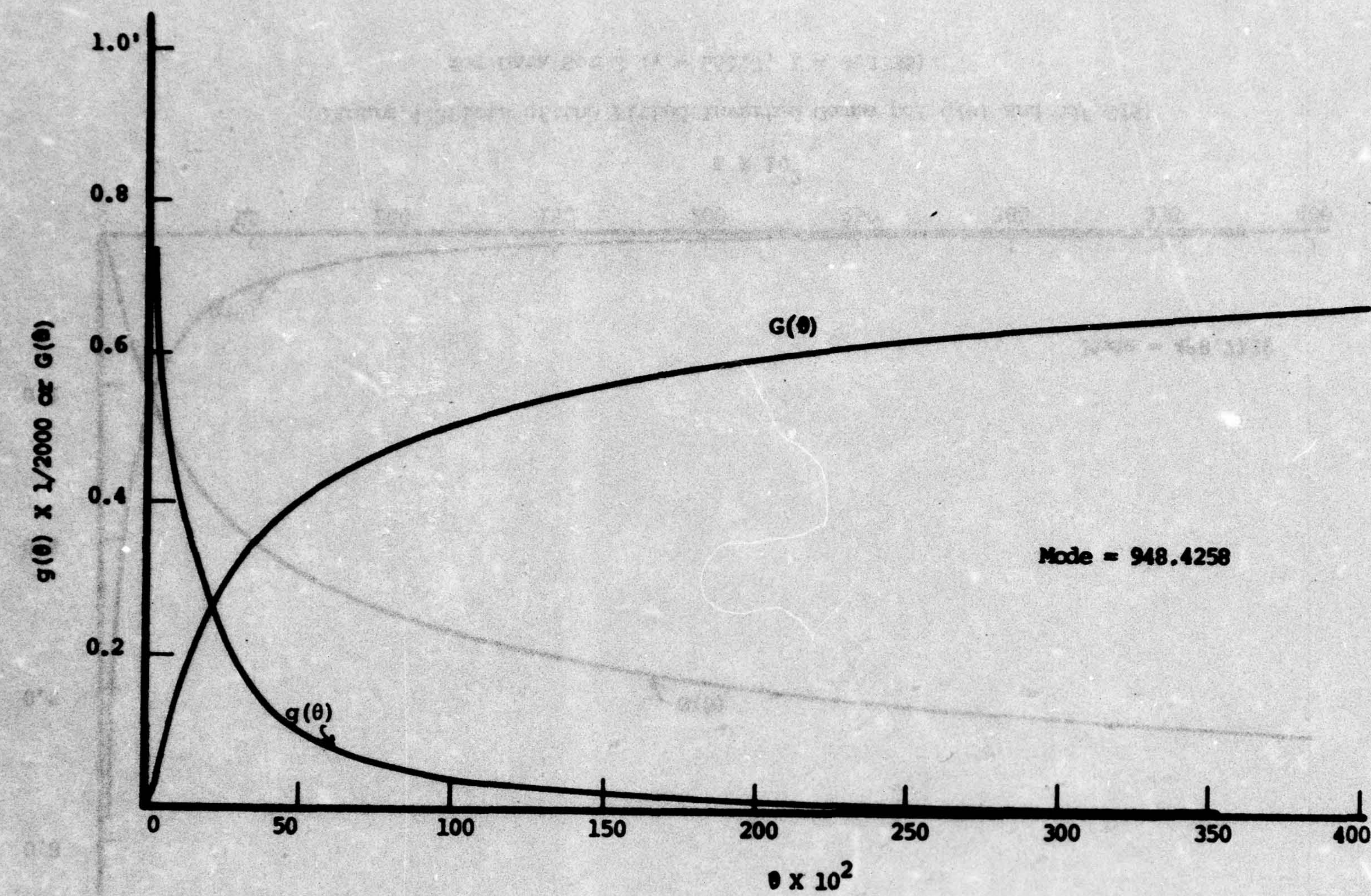

Figure 4.4 Plots of the Fitted Inverted Gamma pdf $g(\theta)$ and cdf $G(\theta)$ for Data Set 4 ($\gamma = 1581.5$, $\lambda = 0.6675$)

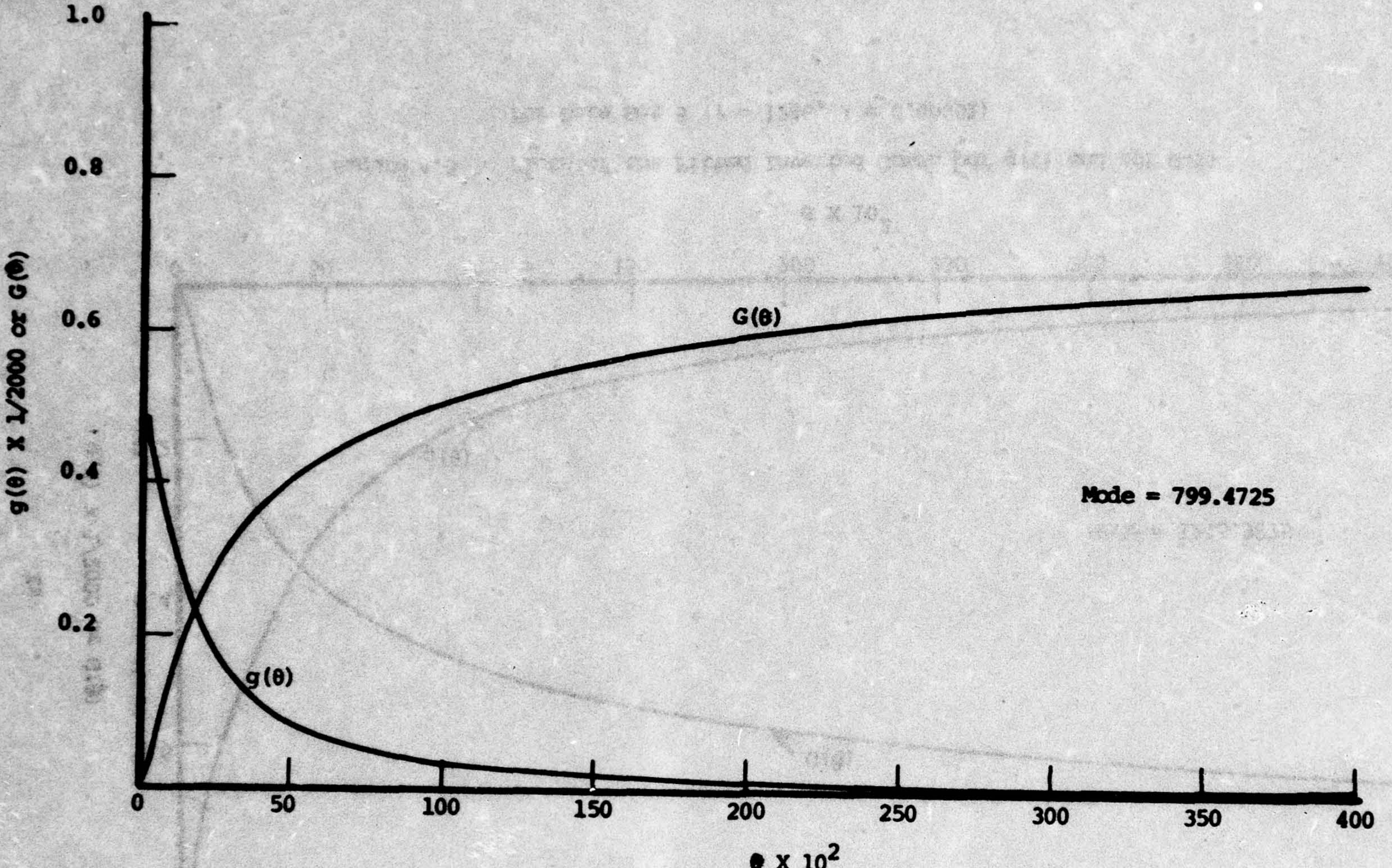

Figure 4.6 Plots of the Fitted Inverted Gamma pdf $g(\theta)$ and cdf $G(\theta)$ for Data Set 6 ($\gamma = 1151.8$, $\lambda = 0.4407$)

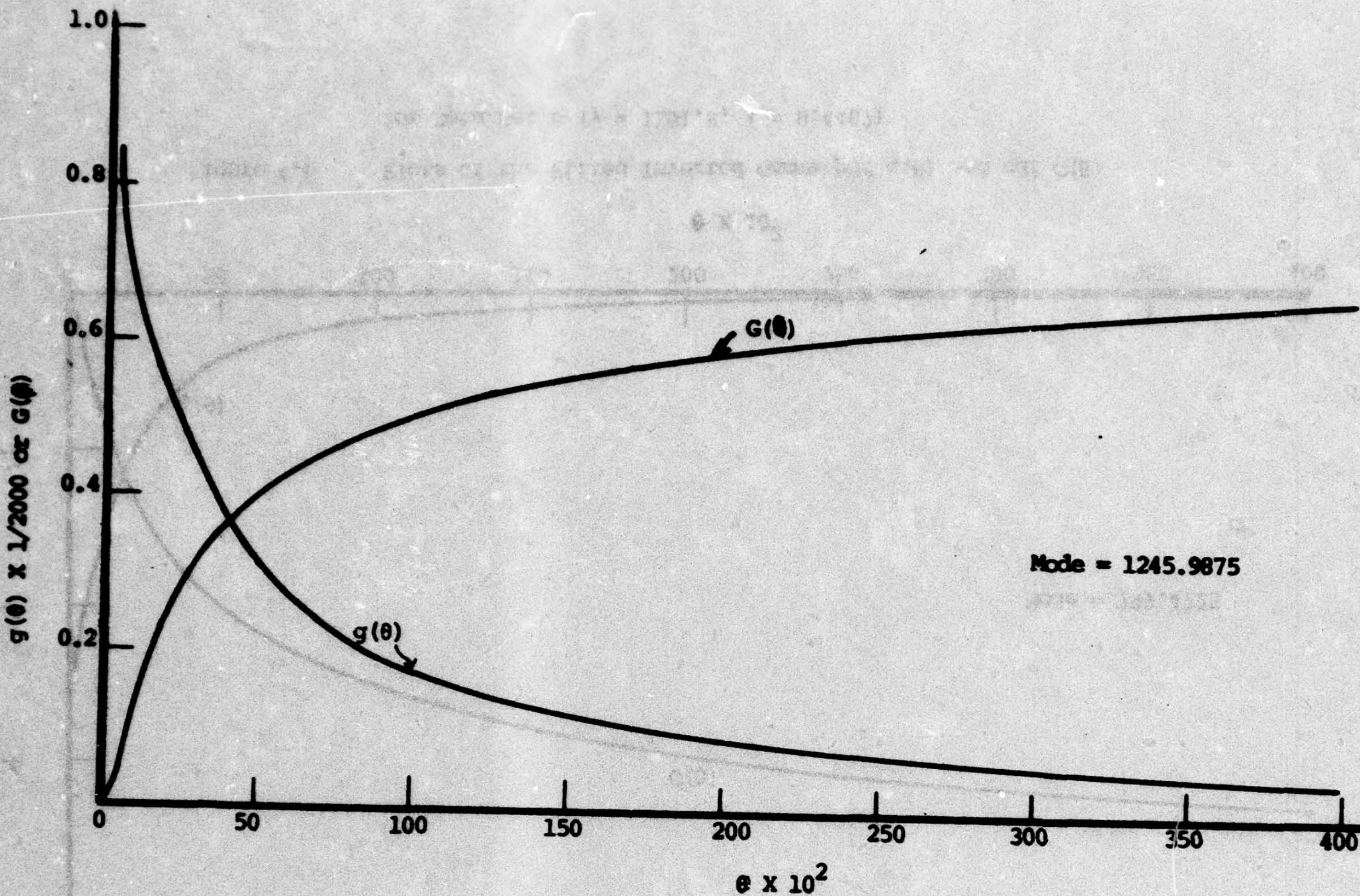

Figure 4.5 Plots of the Fitted Inverted Gamma pdf $g(\theta)$ and cdf $G(\theta)$ for Data Set 5 ($\gamma = 1246$, $\lambda = 0.00001$)

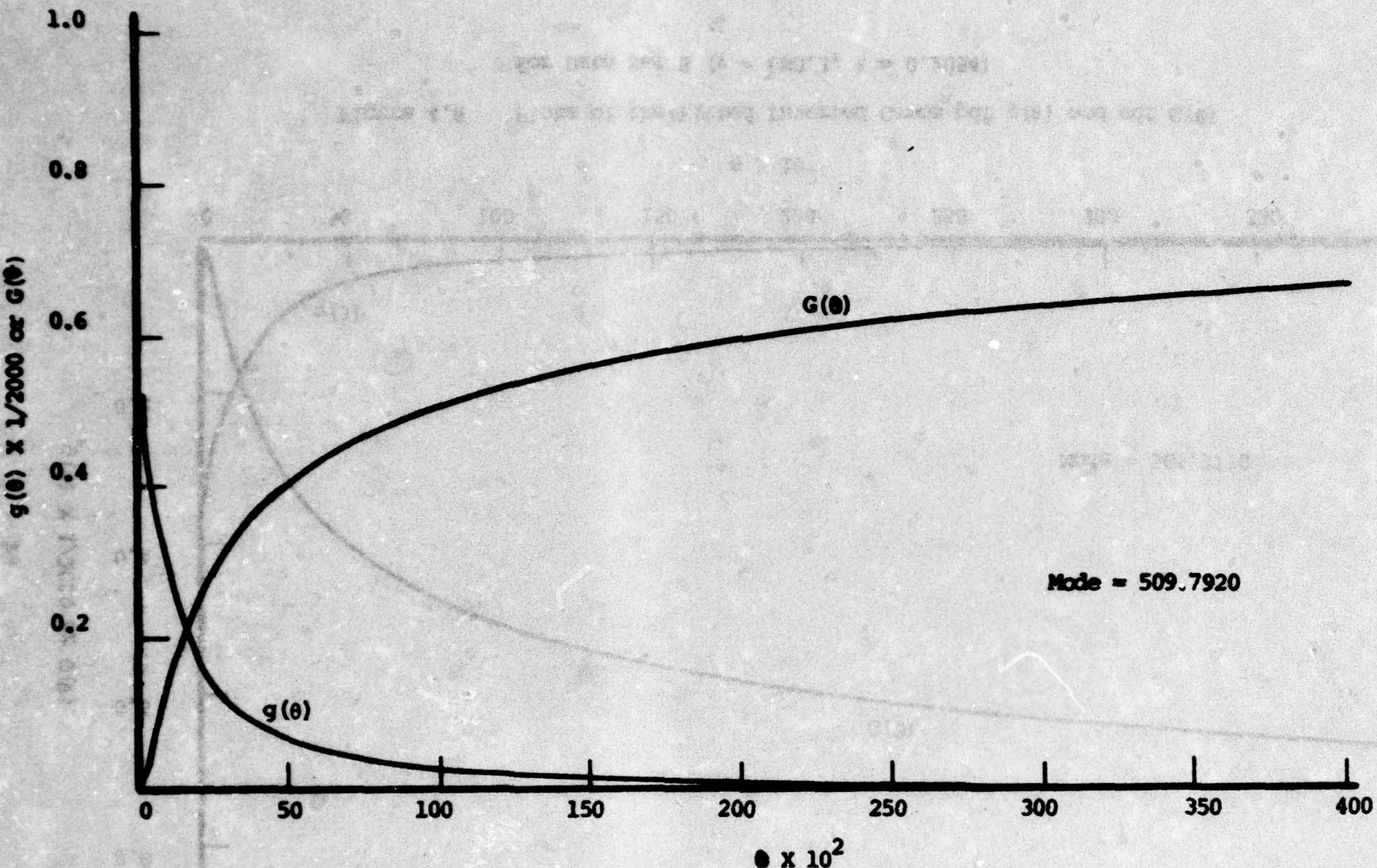

Figure 4.7 Plots of the Fitted Inverted Gamma pdf $g(\theta)$ and cdf $G(\theta)$ for Data Set 7 ($\lambda = 754.9$, $\gamma = 0.4808$)

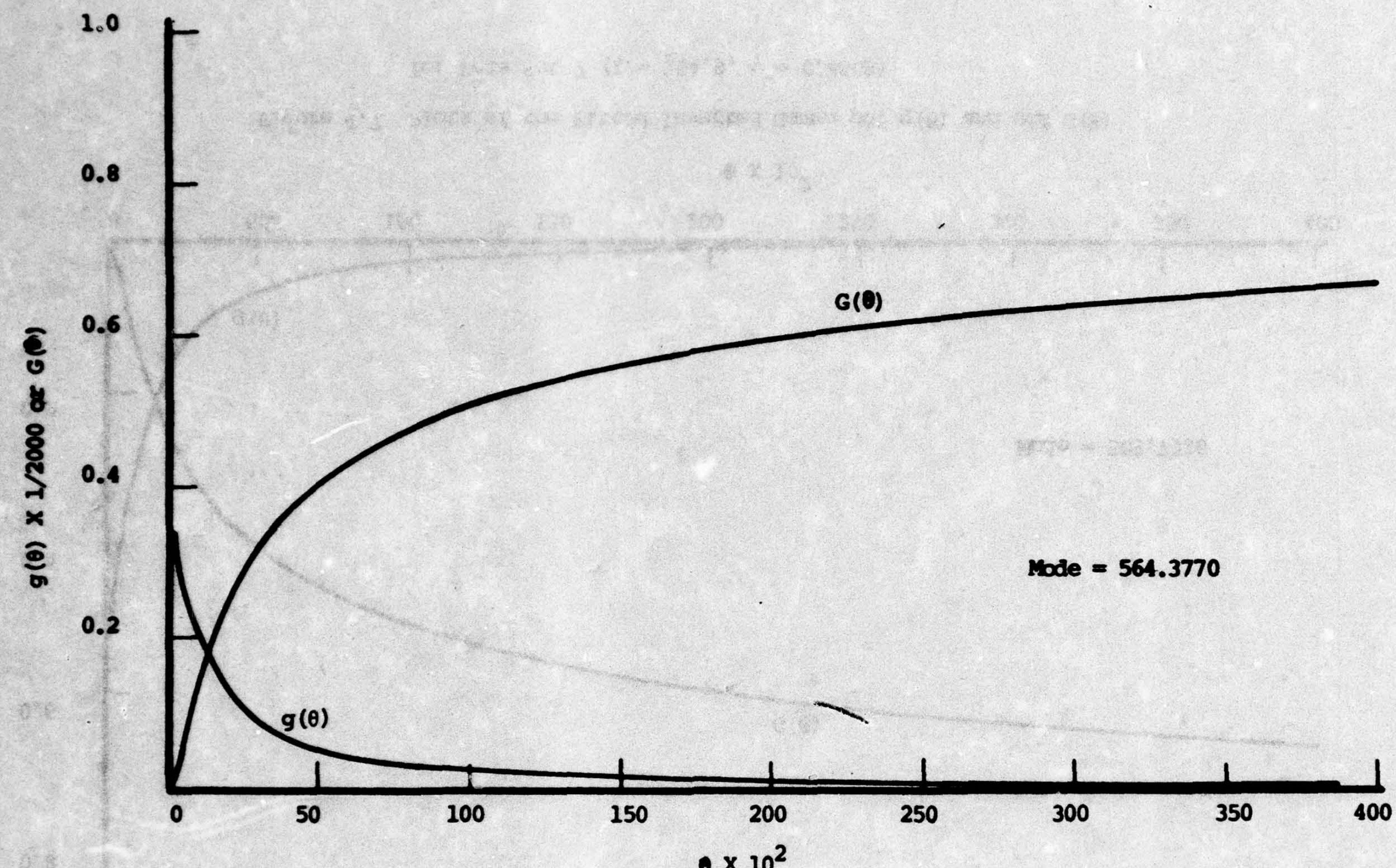

Figure 4.8 Plots of the Fitted Inverted Gamma pdf $g(\theta)$ and cdf $G(\theta)$ for Data Set 8 ($\gamma = 680.3$, $\lambda = 0.2054$)

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RELIABILITY ACCEPTANCE SAMPLING PLANS BASED UPON PRIOR DISTRIBU--ETC(U)
SEP 76 A L GOEL, A M JOGLEKAR F30602-71-C-0312
UNCLASSIFIED TR-76-3 RADC-TR-76-294-VOL-3 NL

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6. CONCLUDING REMARKS

In this report we presented a discussion of the implications of the existence of a prior distribution for the parameter of the exponential failure distribution. Two models were discussed to describe the heterogeneity in the product.

The case when the parameter θ has an inverted gamma density was discussed in detail. The method of obtaining maximum likelihood estimates for the parameters of the inverted gamma distribution was discussed for the case when the data are available for the numbers of failure in fixed testing times. Two cases were discussed: the complete negative binomial and the truncated negative binomial.

Eight sets of field failure data from Schafer (1970) were analyzed in detail to obtain the appropriate fitted distribution and the results were summarized in Table 5.1.

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APPENDIX A

SUBROUTINE TNBINO

PURPOSE AND PROCEDURE:

The purpose of this SUBROUTINE is to fit a truncated negative binomial distribution to the observed failure data.

For the given data, the initial parametric estimates are obtained using the method due to Brass. Then the maximum likelihood estimates are obtained based on the expressions given in Section 4.2 of this report. Also, the variance-covariance matrix of the estimated parameters is calculated using the expressions of Section 4.2.

USAGE:

CALL TNBINO (N,T,F)

PARAMETERS

N - maximum number of failures in the data set
T - testing time
F - input vector of failure frequencies (vector lf length N)

OUTPUT:

Initial estimates of Gamma and Lamda.

Final estimates of Gamma and Lamda

Variance-covariance matrix of Gamma - Lamda

Correlation coefficient between Lamda and Gamma

Table of observed frequencies, expected frequencies based on initial estimates and expected frequencies based on final estimates.

LISTING:

A listing of the SUBROUTINE is given on the following page.

```
      SUBROUTINE TNBINO(N,T,F)
C***** TRUNCATED NEGATIVE BINOMIAL
      DIMENSION X(50),F(50),FI(50),AX(50),FFI(50)
      DIMENSION YYY(500),ZZZ(500)
      DO 900 IWU=1,10
      NOI = 1
      DO 500 LBJ = 1,NOI
      PRINT 200
  200 FORMAT(1H1)
  300 FORMAT(/)
      SF = 0.
      SFX = 0.
      SQ = 0.
      DO 350 I = 1,N
  350 X(I) = I
C****COMPUTE XBAR AND SQR
      DO 2 I = 1,N
    2 SF = SF+F(I)
      DO 3 I = 1,N
    3 SFX = SFX+X(I)*F(I)
      XBAR = SFX/SF
      DO 4 I = 1,N
    4 SQ = SQ+F(I)*(X(I)-XBAR)**2
      SQR = SQ/(SF-1.0)
      PRINT 101,XBAR,SQR
  101 FORMAT(2X,6HXBAR =,F10.4,5X,6HS**2 =,F10.4/)
C****COMPUTE INITIAL ESTIMATES BY METHOD DUE  TO BRASS
      Z = XBAR*T*(SF-F(1))/(SF*SQR-XBAR*(SF-F(1)))
      ZZ = (XBAR*(Z/(T+Z))-(F(1)/SF))/(1.-(Z/(T+Z)))
      PRINT 25,Z,ZZ
   25 FORMAT(2X,7HGAMMA =,F20.5,5X,8HLAMBDA =,F10.5,/)
      FI(1) = (ZZ*((Z/(T+Z))**ZZ)*(T/(T+Z)))/(1.-((Z/(T+Z))**ZZ))
      FI(1) = SF*FI(1)
      DO 20 I = 2,N
   20 FI(I) = ((ZZ+I-1)/I)*(T/(T+Z))*FI(I-1)
C****MAXIMUM LIKELIHOOD ESTIMATES
      Y = Z
      Z = ZZ
      AX(1) = SF
      DO 5 I = 2,N
    5 AX(I) = AX(I-1)-F(I-1)
      DO 400 JFK = 1,20
```

```
      DO 400 JFK = 1,20
      S = 0.
      SS = 0.
      DO 6 I = 1,N
      S = S+(1./ (Z+I-1))*AX(I)
    6 SS = SS+(1./((Z+I-1)**2))*AX(I)
      D = Y/(T+Y)
      E = 1.-D
      DZ = D**Z
      A = SF*Z*(E/Y)/(1.-DZ)-SFX/(T+Y)
      B = SF*ALOG(D)/(1.-DZ)+S
      CC = SF*(E/Y)*(1.-DZ*(1.-Z*ALOG(D)))/((1.-DZ)**2)
      AA = SFX/((T+Y)**2)-(SF*Z*T/(Y*Y*(T+Y)*(T+Y)))*(T+2.*Y-DZ*
     1(T+2.*Y+Z*T))/((1.-DZ)**2)
      BB = SF*DZ*((ALOG(D))**2)/((1.-DZ)**2)-SS
      GG = AA*BB-CC*CC
      PRINT 7,Z,A,B,AA,BB,CC,GG
    7 FORMAT(2X,8E12.5)
      YD = -(BB/GG)*A+(CC/GG)*B
      ZD = (CC/GG)*A-(AA/GG)*B
      Y = Y+YD
      Z = Z+ZD
      IF(ABS(A)-0.001) 8,8,400
    8 IF(ABS(B)-0.001) 9,9,400
  400 CONTINUE
C****VARIANCE-COVARIANCE MATRIX OF MLE
    9 Y=Y-YD
      Z = Z-ZD
      PRINT 300
      PRINT 25,Y,Z
      YYY(LBJ) = Y
      ZZZ(LBZ) = Z
      VG = -BB/GG
      SVG = SQRT(VG)
      PRINT 12,VG,SVG
   12 FORMAT(2X,19HVARIANCE OF GAMMA =,F20.5,19HSTD.DEV. OF GAMMA =,
     1F20.5,/)
      VL = -AA/GG
      SVL = SQRT(VL)
      PRINT 13,VL,SVL
   13 FORMAT(2X,21HVARIANCE OF LAMBDA = ,F20.5,20HSTD.DEV. OF LAMBDA =
     1F20.5,/)
      VLG = CC/GG
      PRINT 14,VLG
   14 FORMAT(2X,25HCOVARIANCE LAMBDA-GAMMA =,F20.5,/)
```

```
   14 FORMAT(2X,25HCOVARIANCE LAMBDA-GAMMA =,F20.5,/)
      RHO = VLG/SQRT(VL*VG)
      PRINT 15,RHO
   15 FORMAT(2X,5HRHO =,F20.5,/)
C******CALCULATIONS IN TERMS OF MEAN
      XM = Z*T/Y
      PRINT 30,XM
   30 FORMAT(2X,6HMEAN =,F20.5,/)
      XA = Z/(XM+Z)
      AAA = XA*(-SFX/(XM*XM)+SF*(XA**(Z+1.))/((1.-XA**Z)**2))
      CCC = -SF*(XA**(Z+1.))*(1.-XA+ALOG(XA))/((1.-XA**Z)**2)
      BBB = BB+CCC*CCC/AAA-CC*CC/AA
      GGG = AAA*BBB-CCC*CCC
      VM = -BBB/GGG
      SVM = SQRT(VM)
      PRINT 31,VM,SVM
   31 FORMAT(2X,18HVARIANCE OF MEAN =,F20.5,18HSTD.DEV. OF MEAN =,F20.
     1,/)
      VVL = -AAA/GGG
      SVVL = SQRT(VVL)
      PRINT 13,VVL,SVVL
      VLM = CCC/GGG
      PRINT 32,VLM
   32 FORMAT(2X,24HCOVARIANCE LAMBDA-MEAN =,F20.5,/)
      RRHO = VLM/SQRT(VVL*VM)
      PRINT 15,RRHO
C*********************************************************************
      FFI(1) = SF*Z*DZ*E/(1.-DZ)
      DO 16 I = 2,N
   16 FFI(I) = ((Z+I-1)/I)*E*FFI(I-1)
   17 FORMAT(5X,1HX,12X,1HF,14X,2HFI,14X,2HFI)
      DO 18 I = 1,N
      PRINT 19,X(I),F(I),FI(I),FFI(I)
   19 FORMAT(3X,F4.0,8X,F5.0,5X,F12.4,5X,F12.4)
   18 CONTINUE
  500 CONTINUE
  900 CONTINUE
      RETURN
      END
```

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